Errata

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Errata

Changes appear in yellow. Line $k+$ (resp., line $k-$) denotes the kth line from the top (resp., the bottom) of a page. My thanks go to the following individuals who have contributed to this list: Tobias Wöhrer, Simon Becker, Dennis Cutraro, Mateusz Piorkowski, Laura Kanzler, Mateus Sampaio, Laura Shou, Noema Nicolussi, Andreas Geyer-Schulz, Rene Allerstorfer, Manuel Culqui Rodriguez, Fritz Gesztesy, Marcel Griesemer, Michael Hofacker, Maxim Zinchenko, Jannik Pitt, Jake Fillman, David Damanik, Peter Stahlecker, Iryna Karpenko, Michael Putzenlechner, Jacob Shapiro, Vedran Sohinger, Mischa Elkner, Stephan Schneider, Cyril Labbé, Chuning Wang.

Page 12. Line after (0.16): Note that $x \in \overline{Y}$ if and only if $\operatorname{dist}(x, Y) = 0$.

Page 16. First line: for $a \in \ell^p(\mathbb{N})$, $b \in \ell^q(\mathbb{N})$.

Page 25. Proof of Theorem 0.25: and we can choose $m_2 = \sqrt{\sum_j \|u_j\|_1^2}$.

Page 34. Proof of Lemma 0.36: (if $K_2(x, .)f(.) \notin L^p(Y, d\nu)$, the inequality is trivially true).

Page 36. Add the following at the end of Lemma 0.39: Moreover, if u and f both have compact support, then $f_k \in C_c^\infty(\mathbb{R}^n)$.

Page 36. Proof of Lemma 0.41: … $\varphi_n \in C_c^\infty(\mathbb{R}^n)$ with support inside some open ball X which converges … continuous functions φ_n with support in X which converges to g …

Page 54. Proof of Lemma 1.11: (ii) follows from $\langle \varphi, A^{**}\psi\rangle = \langle A^*\varphi, \psi\rangle = \langle \varphi, A\psi\rangle$.

Page 60. Last sentence in the proof of Theorem 1.16: Since $f - \varepsilon < f_{z_l}$ for all z_l we have $f - \varepsilon < f_\varepsilon$ and we have found a required function.

Page 61. Problem 1.23: Show that the span of $\{(t-z)^{-1} | z \in U\}$ is dense in $C_\infty(\mathbb{R})$.

Page 66. Line after (2.15): measurable function $A : \mathbb{R}^d \to \mathbb{C}$.

Page 66. Line 4 from the bottom: $|(Af)(x)|^2 = |A(x)|^2 |f(x)|^2 \le \|A\|_\infty^2 |f(x)|^2$

Page 72. Line 18+: Clearly we have $\overline{\alpha A} = \alpha \overline{A}$ for $\alpha \in \mathbb{C}\backslash\{0\}$ and $\overline{A + B} = \overline{A} + B$ provided A is closable and B is bounded (Problem 2.8).

Page 75. Proof of Lemma 2.7:

$$\begin{aligned} \|(A - z)\psi\|^2 &= \|(A - x)\psi - \mathrm{i}y\psi\|^2 \\ &= \|(A - x)\psi\|^2 + y^2\|\psi\|^2 \ge y^2\|\psi\|^2, \end{aligned} \tag{2.46}$$

Page 76. Problem 2.8: Suppose that if A is closable and B is bounded. Show that $\overline{\alpha A} = \alpha\overline{A}$ for $\alpha \in \mathbb{C}\backslash\{0\}$ and $\overline{A + B} = \overline{A} + B$.

Page 78. Proof of Lemma 2.11:

$$\mathfrak{D}(\tilde{A}) = \{\psi \in \mathfrak{H}_A | \exists \tilde{\psi} \in \mathfrak{H} : \langle \varphi, \psi\rangle_A = \langle \varphi, \tilde{\psi}\rangle, \forall \varphi \in \mathfrak{D}(A)\} = \mathfrak{H}_A \cap \mathfrak{D}(A^*)$$

as $\mathfrak{D}(A) \subset \mathfrak{H}_A$ is dense and $\langle \varphi, \psi\rangle_A = \langle (A+1)\varphi, \psi\rangle$ for $\varphi \in \mathfrak{D}(A)$, $\psi \in \mathfrak{H}_A$.

Page 82. Proof of Lemma 2.15:

$$\begin{aligned} 2|\operatorname{Re}\langle \varphi, A\psi\rangle| &= \frac{1}{2}|q(\psi + \varphi) - q(\psi - \varphi)| \le \frac{\|q\|}{2}(\|\psi + \varphi\|^2 + \|\psi - \varphi\|^2) \\ &= \|q\|(\|\psi\|^2 + \|\varphi\|^2) \end{aligned}$$

Page 84. End of the example on the top of the page: Moreover, z is an eigenvalue of A if $\mu(A^{-1}(\{z\})) > 0$ and any square integrable function supported on $A^{-1}(\{z\})$ is a corresponding eigenfunction.

Page 87. Proof of Theorem 2.19:

$$f'(\lambda) = -\|(A - E + \lambda)^{-1}\varphi\|^2 \le -f(\lambda)^2$$

Page 88. Problem 2.18: Then so does $A + B$ if $\|B\| < \|A^{-1}\|^{-1}$.

Page 93. Paragraph after Lemma 2.28: A conjugate linear map $C : \mathfrak{H} \to \mathfrak{H}$ is called a **conjugation** if it satisfies $C^2 = \mathbb{I}$ and $\langle C\psi, C\varphi\rangle = \langle \varphi, \psi\rangle$.

Page 94. Proof of Lemma 2.30:

$$\sum_j \langle \varphi_j(z), .\rangle \psi_j(z^*) = \sum_j \langle \psi_j(z), .\rangle \varphi_j(z^*).$$

Page 114. Proof of Lemma 3.10: The claim is immediate except for measurability of f^{-1}. This is not hard to see if f is strictly monotone. The general case is a nontrivial result which can be found in Theorem 15.1 of [A.S. Kechris, *Classical Descriptive Set Theory*, Springer, New York, 1995].

Page 114. Example after Lemma 3.10: Note that we can still get a unitary map $L^2([0,\infty), d(f_\star\mu)) \oplus L^2((0,\infty), d(f_\star\mu)) \to L^2(\mathbb{R}, d\mu)$, $(\psi_1, \psi_2) \mapsto \psi_1(\lambda^2) + \psi_2(\lambda^2)(\chi_{(0,\infty)}(\lambda) - \chi_{(0,\infty)}(-\lambda))$ (with the convention $\psi_2(0) = 0$).

Page 113. Proof of Lemma 3.11: Then there exists an $\varepsilon > 0$ such that $(\lambda - \varepsilon, \lambda + \varepsilon)$ is disjoint from $\overline{f(\sigma(A))}$ and thus $\|(f(A) - \lambda)^{-1}\| \le \varepsilon^{-1}$,

Page 116. Proof of Lemma 3.13: To see the converse, note that by Theorem 3.27, the set M is a support for μ.

Page 120.

$$C(z) = -\mathrm{i}F(\mathrm{i}\frac{1-z}{1+z}). \tag{3.91}$$

Page 126. Proof of Theorem 3.26: Clearly the second part can be estimated by

$$\int_{\mathbb{R}\setminus I_\delta} K_\varepsilon(t-\lambda) d\mu(t) \le \varepsilon \int_{\mathbb{R}\setminus I_\delta} \frac{d\mu(t)}{(t-\lambda)^2}.$$

Page 127. Proof of Lemma 3.30: First of all, note that we can split $F(z) = F_1(z) + F_2(z)$ according to $d\mu = \chi_{[-1,1]} d\mu + (1 - \chi_{[-1,1]}) d\mu$.

Page 135. Problem 4.11:

$$\chi_\Omega(A) = -\frac{1}{2\pi\mathrm{i}} \int_\Gamma R_A(z) dz,$$

Page 138. Equation (4.31) and the text before it should be changed according to (the original argument only works on $\mathfrak{D}(A^*A)$ and not on $\mathfrak{D}(A)$):
Since $A^*A = |A|^2$ the same is true for the associated quadratic forms, that is,

$$\||A|\psi\|^2 = \|A\psi\|^2, \quad \psi \in \mathfrak{D}(|A|) = \mathfrak{D}(A), \tag{4.31}$$

Page 139:

$$U^*U = P_{\mathrm{Ker}(A)^\perp}, \qquad UU^* = P_{\mathrm{Ker}(A^*)^\perp}, \tag{4.34}$$

Page 139: Last line of Theorem 4.10: $\mathrm{Ker}(U)^{\perp} = \mathrm{Ker}(A)^{\perp}$

Page 141: (ii) We have

$$\inf_{\psi \in U(\varphi_1,\dots,\varphi_{n-1})} \langle \psi, A\psi \rangle \geq E_n, \tag{4.40}$$

since A restricted to $\mathrm{span}\{\varphi_1, \dots, \varphi_{n-1}\}^{\perp}$ is bounded from below by E_n (which is immediate from the spectral theorem).

Page 141: Corollary 4.13: Suppose A and B are self-adjoint operators with $\mathfrak{D}(A) = \mathfrak{D}(B)$ and $A \geq B$ (i.e., $A - B \geq 0$).

Page 143: Proof of Theorem 4.16: Thus $\langle \psi, (A-\lambda_2)(A-E)\psi \rangle = \|A\psi\|^2 + \lambda_2 E \geq 0$ and ...

Page 144: Proof of Theorem 4.17: Consequently, $(P - \lambda)\psi_k \to 0$, where $\psi_k = \psi_{1,k} \otimes \cdots \otimes \psi_{n,k}$ and hence $\lambda \in \sigma(P)$.

Page 146: Proof of Lemma 5.2:

$$\frac{d}{dt}\varphi(t) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \Big(\big(U(-t-\varepsilon) - U(-t)\big)\big(\psi(t) - \varepsilon \mathrm{i} A\psi(t) + o(\varepsilon)\big) + U(-t)\big(\psi(t+\varepsilon) - \psi(t)\big) \Big) = 0.$$

Page 148: Top of page: Let $\psi \in \mathfrak{D}(A)$ and abbreviate $\psi(t) = (U(t) - V(t))\psi$. Then

$$\lim_{\varepsilon \to 0} \frac{\psi(t+\varepsilon) - \psi(t)}{\varepsilon} = -\mathrm{i}\overline{A}\psi(t)$$

and hence $\frac{d}{dt}\|\psi(t)\|^2 = -2\,\mathrm{Re}\langle \psi(t), \mathrm{i}\overline{A}\psi(t) \rangle = 0$.

Page 152: Proof of Theorem 5.7: Since $K(A-\mathrm{i})^{-1}$ is compact by assumption,

Page 154: Proof of Theorem 5.9: We will assume that K is compact.

Page 155. Problem 5.7:

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T \langle \varphi, \mathrm{e}^{\mathrm{i}t(A-\lambda_0)}\psi \rangle dt = \langle \varphi, P_A(\{\lambda_0\})\psi \rangle$$

Page 155. Problem 5.9:

$$\mathfrak{H}_{rc} = \{\psi \in \mathfrak{H} | \lim_{t \to \infty} \langle \psi, \mathrm{e}^{-\mathrm{i}tA}\psi \rangle = 0\} \supseteq \mathfrak{H}_{ac}, \tag{5.27}$$

Page 156. Proof of Theorem 5.11:

$$\big\| \big((\mathrm{e}^{\mathrm{i}\tau A}\mathrm{e}^{\mathrm{i}\tau B})^n - \mathrm{e}^{\mathrm{i}t(A+B)} \big)\psi \big\| \leq |t| \max_{|s| \leq |t|} F_\tau(s),$$

Page 159. Theorem 6.4:

$$\gamma - \max\Big(a|\gamma| + b, \frac{b}{1-a}\Big). \tag{6.4}$$

Page 159. Proof of Theorem 6.4: By the above lemma, we can find a $\lambda > 0$ such that $\|BR_A(\mp\mathrm{i}\lambda)\| < 1$. Hence $-1 \in \rho(BR_A(\mp\mathrm{i}\lambda))$ and thus $\mathbb{I} + BR_A(\mp\mathrm{i}\lambda)$ is onto. Thus

$$(A + B \pm \mathrm{i}\lambda) = (\mathbb{I} + BR_A(\mp\mathrm{i}\lambda))(A \pm \mathrm{i}\lambda)$$

is onto and the proof of the first part is complete.

Page 159. Proof of Theorem 6.4; last sentence: The explicit bound (6.4) follows since this condition implies $\|BR_A(-\lambda)\| < 1$ by virtue of (6.2) from the proof of the previous lemma.

Page 161. Lemma 6.8:

$$s_n(K) = \min_{\psi_1,\dots,\psi_{n-1}} \sup_{\psi\in U(\psi_1,\dots,\psi_{n-1})} \|K\psi\|, \tag{6.9}$$

Page 162. Proof of Lemma 6.9: last formula

$$\gamma_n = \|K - K_n\| = \sup_{\|\psi\|=1} \|K(\psi - \sum_{j=1}^{n} \langle\varphi_j, \psi\rangle \varphi_j)\|$$

Page 164. Proof of Lemma 6.10: Conversely, choose $\varphi_i = \hat{\phi}_i$

Page 172. Proof of Lemma 6.21: The proof of the stated formula only requires the first resolvent identity.

Page 176. Theorem 6.25: add for $\lambda > \frac{b}{a} - \gamma$. after (6.44)

Page 176. Line before equation (6.45): Furthermore, we can define $C_q(\lambda)$ for all $-z \in \rho(A)$, using

Page 177f. Lemmas 6.28, 6.29, and 6.30: Suppose $A - \gamma \ge 0$ and B are self-adjoint.

Page 179. Problem 6.18: Suppose A is self-adjoint, $\lambda \in \mathbb{R}$, and R is bounded. Show that $R = R_A(\lambda)$ if and only if $\langle (A-\lambda)\varphi, R\psi\rangle = \langle\varphi,\psi\rangle$ for all $\varphi \in \mathfrak{D}(A)$, $\psi \in \mathfrak{H}$.

Page 180. Corollary 6.32: Then this holds for all z in the interior of Γ.

Page 181. Proof of Lemma 6.37

$$\begin{aligned}\|R_{A_n}(z)\psi\|^2 - \|R_A(z)\psi\|^2 &= \langle\psi, R_{A_n}(z^*)R_{A_n}(z)\psi - R_A(z^*)R_A(z)\psi\rangle\\ &= \frac{1}{z-z^*}\langle\psi, (R_{A_n}(z) - R_{A_n}(z^*) - R_A(z) + R_A(z^*))\psi\rangle \to 0.\end{aligned}$$

Page 182. Proof of Theorem 6.38, penultimate line: ..., which implies $\|R_{A_n}(\lambda_n + \mathrm{i})\| < 1$ for n sufficiently large, ...

Page 195. Line 2+: Clearly $H^{r+1}(\mathbb{R}^n) \subset H^r(\mathbb{R}^n)$

Page 200. Discussion after Lemma 7.20: $|\psi(x,t)|^2 d^n x = |\hat{\psi}(\frac{x}{2t})|^2 \frac{d^n x}{(2t)^n}$

Page 209. Proof of Theorem 8.2:

$$|\langle \hat{A}\psi, \hat{B}\psi\rangle|^2 = |\langle \psi, \hat{A}\hat{B}\psi\rangle|^2 = \frac{1}{4}|\langle \psi, \{\hat{A}, \hat{B}\}\psi\rangle|^2 + \frac{1}{4}|\langle \psi, [A,B]\psi\rangle|^2,$$

Page 209. Last line of the proof of Theorem 8.2: $0 = (z + z^*)\|\hat{A}\psi\|^2$

Page 209.

$$\psi(x) = \left(\frac{\lambda}{\pi}\right)^{n/4} \mathrm{e}^{-\frac{\lambda}{2}|x-x_0|^2 + \mathrm{i}p_0 x}, \tag{8.13}$$

Page 211. First equation in the proof of Lemma 8.3:

$$\frac{1}{\sqrt{2\pi}} \int \overline{\varphi(x)} \mathrm{e}^{-\frac{x^2}{2}} \sum_{j=0}^{k} \frac{(\mathrm{i}tx)^j}{j!} dx = 0$$

Page 212. Theorem 8.4: There exists an orthonormal basis of simultaneous eigenvectors for the operators L^2 and L_3.

Page 213. Line before (8.41): If $N\psi_0 = 0$, then we must have $A_-\psi_0 = 0$, and the normalized solution of this last equation is given by

Page 215. Proof of Theorem 8.6:

$$\begin{aligned} R_{H_1}(z) &= R_{H_1}(z)(1 - P_1) - \frac{1}{z} P_1 = U R_{H_0}(z) U^* - \frac{1}{z} P_1 \\ &\supseteq \frac{1}{z} \left(U(|H_0|^{1/2} R_{H_0}(z) |H_0|^{1/2} - 1) U^* - P_1 \right) \\ &= \frac{1}{z} \left(A R_{H_0}(z) A^* - (1 - P_1) - P_1 \right) = \frac{1}{z} \left(A R_{H_0}(z) A^* - 1 \right), \end{aligned}$$

Page 222. Proof of Lemma 9.5: Choosing $f_1 = v$, $f_2 = f$, $f_3 = v^*$, $f_4 = f^*$, we infer (9.15).

Page 222. Problem 9.1: and $f(d) = \gamma$, $(pf')(d) = \delta$.

Page 222. Problem 9.3: Let $\phi \in L^1_{loc}(I)$ be real-valued.

Page 222. Problem 9.4: Add the assumption that a is regular. Otherwise one can also start the integration at an arbitrary point in (a,b).

Page 223. Problem 9.4: where

$$Q = \frac{q}{r} - \frac{(pr)^{1/4}}{r}\big(p((pr)^{-1/4})'\big)'.$$

Page 223. Replace the last sentence by: Moreover, the following set is a core for A

$$(9.21)\qquad \mathfrak{D}_1 = \{f \in \mathfrak{D}(\tau)|\ \exists x_0 \in I:\ \forall x \in (a, x_0),\ V_x(f) = 0,\\ \exists x_1 \in I:\ \forall x \in (x_1, b),\ W_x(f) = 0\},$$

where we set $V_x(f) = W_x(v, f)$, $W_x(f) = W_x(w, f)$ if τ is l.c. at a, b and $V_x(f) = f(x)$, $W_x(f) = f(x)$ if τ is l.p. at a, b, respectively.

Page 224.

$$(9.23)\qquad W_a(v, f) = 0 \quad \Leftrightarrow \quad \cos(\alpha)BC_a^2(f) + \sin(\alpha)BC_a^1(f) = 0,$$

where $\tan(\alpha) = -\frac{BC_a^2(v)}{BC_a^1(v)}$.

Page 228. Theorem 9.10: Delete "(which are simple)". And the following claim about simplicity of eigenvalues only applies to separated boundary conditions as in Theorem 9.6.

Page 231.

$$(9.37)\qquad (Uf)(\lambda) = \frac{1}{\sqrt{2\pi}}\begin{pmatrix} \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}\sqrt{\lambda}x} f(x)\,dx \\ \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}\sqrt{\lambda}x} f(x)\,dx \end{pmatrix}, \quad \lambda \in \sigma(H_0) = [0, \infty).$$

Page 233. Proof of Lemma 9.13:

$$\sum_j \int_{\mathbb{R}} F_j(\lambda)^* \int_a^b u_j(\lambda, x) g(x) r(x)\,dx\,d\mu_j(\lambda) = \int_a^b (U^{-1}\underline{F})(x)^* g(x) r(x)\,dx.$$

Interchanging integrals on the left-hand side

Page 233. Delete the last sentence: ~~Note that since we can replace $u_j(\lambda, x)$ by $\gamma_j(\lambda)u_j(\lambda, x)$ where $|\gamma_j(\lambda)| = 1$, it is no restriction to assume that $u_j(\lambda, x)$ is real-valued.~~

Page 250. Second line in Section 9.7: on $(a, b) = \mathbb{R}$.

Page 252. Proof of Lemma 9.35: where $M_n = \sup_{|m|\geq n} \int_m^{m+1} |q(x)| dx$.

Page 255. First line: the zeros of ψ_n interlace the zeros of ψ_{n+1}.

Page 256. Problem 9.18: Change the hint according to:
(Hint: Let $\varphi_\varepsilon(x) = \exp(-\varepsilon^2 x^2)$ and investigate $\langle \varphi_\varepsilon, H\varphi_\varepsilon\rangle$.)

Page 261.
(10.23)

$$A\Phi = \tau\Phi, \quad \mathfrak{D}(A) = \{\Phi \in L^2(0,2\pi) | \ \Phi \in AC^1[0,2\pi], \Phi'' \in L^2(0,2\pi), \\ \Phi(0) = \Phi(2\pi), \Phi'(0) = \Phi'(2\pi)\}.$$

Page 268. Line 3+: Note that the $L_j^{(k)}(r)$ are polynomials of degree j which

Page 321. Lemma A.29: If S_j generates Σ_j and $X_j \in S_j$ for $j = 1, 2$, then

Page 322.

$$F(z) = \int_Y f(z,y)\, d\mu(y) \tag{A.55}$$

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$$\pi^{n/2} = \int_{\mathbb{R}^n} \mathrm{e}^{-|x|^2} d^n x = nV_n \int_0^\infty \mathrm{e}^{-r^2} r^{n-1} dr = \frac{nV_n}{2} \int_0^\infty \mathrm{e}^{-s} s^{n/2-1} ds \\ = \frac{nV_n}{2} \Gamma(\frac{n}{2}) = V_n \Gamma(\frac{n}{2}+1)$$

Page 330. Proof of Lemma A.36: Now set $\varphi(x) = 1$ for $|x| \le R$, $\varphi(x) = 1 - |x| - R$ for $R \le |x| \le R+1$, and $\varphi(x) = 0$ for $|x| \ge R+1$.

Page 330. Problem A.32 can be deleted as the claim is part of Lemma A.36.

Page 333. Problem A.33: Show that the Radon–Nikodym derivative with respect to Lebesgue measure equals the ordinary derivative $\mu'(x)$.

Page 333. Problem A.34: This claim is clearly wrong (take a function which is constant on an interval). It should be deleted.

Page 334. Example:

$$(D\mu)(x) = \lim_{\varepsilon\downarrow 0} \frac{\mu((x-\varepsilon, x+\varepsilon))}{2\varepsilon} = \lim_{\varepsilon\downarrow 0} \frac{\mu(x+\varepsilon) - \mu(x-\varepsilon)}{2\varepsilon} = \mu'(x).$$

Page 338. Proof of Lemma A.47.

$$|M_\varepsilon| = \int_{M_\varepsilon} d^n x \le \frac{1}{\varepsilon} \int_{M_\varepsilon} (D\mu)(x) d^n x = \frac{1}{\varepsilon} \mu_{ac}(M_\varepsilon) \le \frac{1}{\varepsilon} \mu_{ac}(M_0) = 0$$

Addendum

Page 81. Proof of Theorem 2.14: Since the rest is not so straightforward, here is a complete proof:

Proof. Since $\mathfrak{H}_q$ is dense, $\tilde{\psi}$ and hence A is well-defined. Moreover, replacing q by $q(.)-\gamma\|.\|^2$ and A by $A-\gamma$, it is no restriction to assume $\gamma=0$. Next it will be convenient to look at the definition from a somewhat more abstract point of view: We have a conjugate linear continuous embedding $j:\mathfrak{H}\to\mathfrak{H}_q^*$, $\psi\mapsto\langle\psi,.\rangle$ (here $\mathfrak{H}_q$ is equipped with $\|.\|_q$) with $\mathrm{Ran}(j)$ dense. Indeed, if $\mathrm{Ran}(j)$ were not dense, there would be some nonzero $\varphi\in\mathfrak{H}_q^{**}\cong\mathfrak{H}_q$ (the identification given by the Riesz lemma via evaluation) such that $\varphi(j(\psi))=j(\psi)(\varphi)=\langle\psi,\varphi\rangle=0$ for all $\psi\in\mathfrak{H}$ implying the contradiction $\varphi=0$.

Next, there is a conjugate linear isometric isomorphism $\hat{A}:\mathfrak{H}_q\to\mathfrak{H}_q^*$, $\psi\mapsto s(\psi,.)+\langle\psi,.\rangle$ (Riesz lemma) and our operator A is given by $j^{-1}\hat{A}-\mathbb{I}$. Moreover, $\mathfrak{D}(A)=\hat{A}^{-1}\,\mathrm{Ran}(j)$ is dense in $\mathfrak{H}_q$ and hence also in $\mathfrak{H}$. By construction, $q_A(\psi)=q(\psi)$ for $\psi\in\mathfrak{D}(A)$, which shows that A is nonnegative and from $\mathrm{Ran}(A+1)=\mathrm{Ran}(j^{-1}\hat{A})=\mathfrak{H}$ we conclude that A is self-adjoint. Finally, note that the fact that $\mathfrak{D}(A)$ is dense in $\mathfrak{H}_q$ implies $\mathfrak{H}_A=\mathfrak{H}_q$.

Concerning uniqueness let $\tilde{A}$ be another self-adjoint operator with the same properties. Then equality of the associated quadratic forms (and hence of the sesquilinear forms) on $\mathfrak{Q}$ implies $\langle A\psi,\varphi\rangle=\langle\psi,\tilde{A}\varphi\rangle$ for $\psi\in\mathfrak{D}(A)$, $\varphi\in\mathfrak{D}(\tilde{A})$. But this shows $\psi\in\mathfrak{D}(\tilde{A}^*)=\mathfrak{D}(\tilde{A})$ and $\tilde{A}\psi=\tilde{A}^*\psi=A\psi$ and vice versa. □

Page 118. Here is an amplification of Theorem 3.16:

Theorem 3.16. *For every self-adjoint operator A there is an ordered spectral basis $\{\psi_j\}_{j=1}^N$. Moreover, it can be chosen such that $d\mu_{\psi_j}=\chi_{\Omega_j}d\mu$, where μ is a maximal spectral measure and $\Omega_{j+1}\subseteq\Omega_j$. The dimension N is the spectral multiplicity of A.*

Proof. First of all observe that for every φ there is a maximal spectral vector ψ such that $\varphi\in\mathfrak{H}_\psi$. To see this start with a maximal spectral vector $\tilde{\psi}$. Then $d\mu_\varphi=f\,d\mu_{\tilde{\psi}}$ and we set $\Omega=\{\lambda|f(\lambda)>0\}$. Then $P_A(\Omega)\varphi=\varphi$ since $\|P_A(\Omega)\varphi\|^2=\int_\Omega d\mu_\varphi=\int_{\mathbb{R}}f\,d\mu_{\tilde{\psi}}=\|\varphi\|^2$. Now set $\psi=\varphi+P(\mathbb{R}\backslash\Omega)\tilde{\psi}$ and observe $d\mu_\psi=d\mu_\varphi+\chi_{\mathbb{R}\backslash\Omega}d\mu_{\tilde{\psi}}=(f+\chi_{\mathbb{R}\backslash\Omega})d\mu_{\tilde{\psi}}$. Since $f+\chi_{\mathbb{R}\backslash\Omega}>0$ we see that $d\mu_{\tilde{\psi}}$ is absolutely continuous with respect to $d\mu_\psi$ and hence ψ is a maximal spectral vector with $\varphi=P_A(\Omega)\psi\in\mathfrak{H}_\psi$ as required.

Now start with some total set $\{\tilde{\psi}_j\}$ and proceed as in Lemma 3.4 to obtain an ordered spectral basis $\{\psi_j\}$. Since $\mu_{\psi_{j+1}}$ is absolutely continuous with respect to μ_{ψ_j} all spectral measures are absolutely continuous with respect to $\mu=\mu_{\psi_1}$, that is, $d\mu_{\psi_j}=f_jd\mu$. Choosing $\Omega_j=\{\lambda|f_j(\lambda)>0\}$ we can replace $\psi_j\to\chi_{\Omega_j}(A)f_j(A)^{-1/2}\psi_j$ such that $f_j\to\chi_{\Omega_j}$. Since $\mu_{\psi_{j+1}}$ is absolutely continuous with respect to μ_{ψ_j} we can even assume $\Omega_{j+1}\subseteq\Omega_j$.

Finally, we show that the spectral multiplicity of A is N. By the first part we can assume that A is multiplication by λ in $\bigoplus_{j=1}^N L^2(\mathbb{R},\chi_{\Omega_j}d\mu)$. Let $\{\psi_j\}_{j=1}^n$ be a spectral basis with $n<N$. We will show that there is some vector in the orthogonal complement of $\bigoplus_j\mathfrak{H}_{\psi_j}$. Of course such a vector exists pointwise for every λ but it is not clear that the components can be chosen measurable. To see this we use a Gauss-type elimination: For this note that we can multiply every vector ψ_j with a non-vanishing function or add multiples of the other

vectors to a given one without changing $\bigoplus_j \mathfrak{H}_{\psi_j}$. Hence we can first normalize the first component of every ψ_j to be a characteristic function. Moreover, by adding all other vectors to ψ_1 we can assume that its first component is positive on a maximal set $\tilde{\Omega}_1$. In fact, after another normalization we can assume that $\psi_{1,1} = \chi_{\tilde{\Omega}_1}$ and after subtracting multiples of ψ_1 from the remaining vectors we can assume $\psi_{j,1} = 0$ for $j \geq 2$. If $\mu_1(\mathbb{R}\backslash\tilde{\Omega}_1) > 0$ then $\varphi = (\chi_{\mathbb{R}\backslash\tilde{\Omega}_1}, 0, \dots)$ would be in the orthogonal complement and we are done. So assume $\chi_{\Omega_1} = 1$ and continue with the other components until they satisfy $\psi_{j,k} = \delta_{j,k}$ for $1 \leq j, k \leq n$. Then $\varphi = (-\psi_{1,n+1}, \dots, -\psi_{n,n+1}, 1, 0, \dots)$ is in the orthogonal complement contradicting our assumption that $\{\psi_j\}_{j=1}^n$ is a spectral basis. $\square$