UMBRAL CALCULUS AND THE BOUSTROPHEDON TRANSFORM

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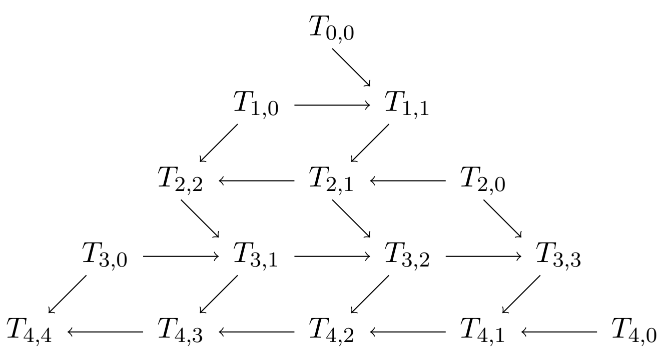
Abstract. The boustrophedon transform is a sequence operation developed in the study of alternating permutations. This paper looks into its construction and explores the relations between the two by developing a bijection between paths on the triangle used in the construction of the transform and alternating permutations on $[n]$.

1. Introduction

The boustrophedon transform of a sequence (a_n) produces a sequence (b_n) by populating a triangle in the following manner:



where the numbers $T_{k,n}$ $(k \geq n \geq 0)$ are defined

$$T_{k,0} = a_k \ (k \geq 0)$$

$$T_{k,n} = T_{k,n-1} + T_{k-1,k-n} \ (k \geq n > 0).$$

Now,

$$b_n = T_{n,n}.$$

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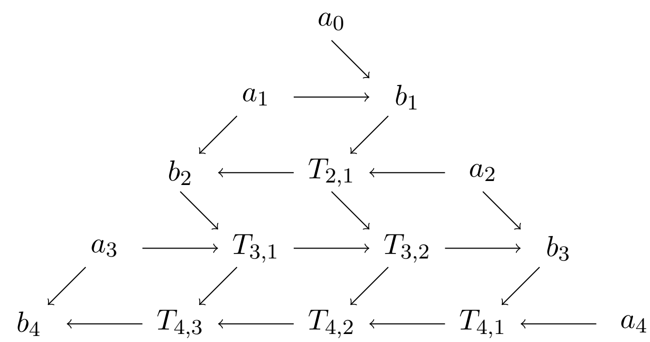
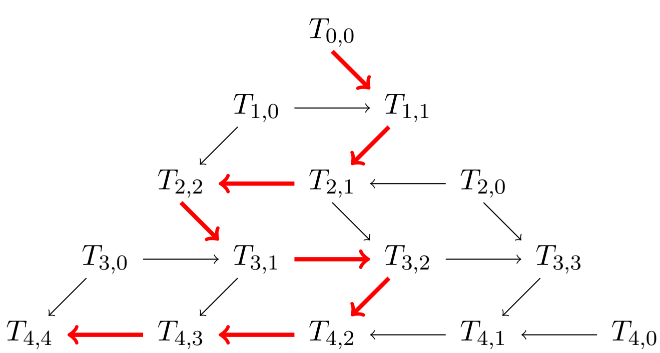
We could rewrite the above triangle as follows:



We begin counting the rows of the triangle from the topmost, zeroth row, downward to the n^{th} row. Let $b_0 = a_0$ populate the top row of the triangle. The odd indexed terms of (a_n) form the left-most entries of the odd rows of the triangle, while the odd indexed terms of (b_n) form the right-most entries of the odd rows. Similarly, the even indexed terms of (a_n) populate the right-most entries of the even rows and the even indexed terms of (b_n) form the left-most entries of the even rows.

Each entry in an odd row is the sum of the term to its left and the term in the preceding row to its upper left. Each even rowed entry is the sum of the term to its right and the term in the preceding row to its upper right. The arrows in the above diagram illustrate the zig-zag pattern of the transform. This is where the boustrophedon moniker originates.

We can view this triangle as a directed graph by taking $\{T_{k,n} \mid k \geq n \geq 0\}$ as the set of vertices and the arrows as directed edges. For example,



shows a path from $T_{0,0}$ to $T_{4,4}$ by the arrows in red. We occasionally denote the vertex $T_{k,n}$ by the ordered pair (k, n). This digraph is called the boustrophedon graph.

The boustrophedon transform developed as a generalization of the Seidel-Entringer-Arnold method of calculating alternating permutations as seen in [1]. An alternating permutation of the set $[n]$ is an arrangement of those numbers into an order $c_1, \ldots, c_n$ such that no element c_i is between c_{i-1} and c_{i+1} for any value of i and $c_1 < c_2$. In other words, $c_i < c_{i+1}$ if i is odd and $c_i > c_{i+1}$ if i is even. We use $DU(n)$ to denote the set of down-up permutations on $[n]$, that is to say, $\sigma = \sigma_1\sigma_2 \ldots \sigma_n$ where $\sigma_i \in [n]$ for all $1 \leq i \leq n$ and with $\sigma_1 > \sigma_2$, $\sigma_2 < \sigma_3$, $\sigma_3 > \sigma_4$, and so forth. In this paper, we show that there is a bijection between the set of all paths from $(0, 0)$ to (n, n) in the boustrophedon graph and the set $DU(n)$ of all down-up permutations on $[n]$, and we further explore the boustrophedon transform by using umbral calculus on a more general transformation of sequences.

2. Examples

We now explore several examples using the boustrophedon transform.

Example 1. The boustrophedon transform of the sequence $(1, 0, 0, 0, \ldots)$ generates the Euler numbers as seen in Figure 1.

1

0 1

1 1 0

0 1 2 2

5 5 4 2 0

Figure 1. Triangle generated by $(1, 0, 0, 0, \ldots)$

Example 2. The boustrophedon transform of the Catalan numbers generates the sequence $(D_n) = (1, 3, 10, 37, 149, 648, 3039, 15401, 84619, 505500, \ldots)$ as shown in Figure 2.

1

2 3

10 8 5

14 24 32 37

149 135 111 79 42

Figure 2. Triangle generated by the Catalan numbers

Example 3. Applying the boustrophedon transform successively to the Euler numbers yields the sequence $(B_n) = (1, 2, 4, 10, 32, 122, 544, 2770, 15872, 101042, \ldots)$ as seen in Figure 3. This sequence is in fact that of the number of zig-zag permutations on n.

$$\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 2 & & & \\
 & & 4 & & 3 & & 1 & & \\
 & 2 & & 6 & & 9 & & 10 & \\
32 & & 30 & & 24 & & 15 & & 5
\end{array}$$

FIGURE 3. Triangle generated by the Euler numbers

Example 4. The sequence of Euler numbers, or up/down numbers, is given by the EGF, $E(x) = \tan x + \sec x$, with the first few terms being 1, 1, 1, 2, 5, 16, 61, 272, 1385, 7936, 50521. These numbers are notable for $n \geq 2$; they are half the number of alternating permutations on $[n]$.

Example 5. Taking $b_n = \sum_{k=0}^{n} \binom{n}{k} a_k$ to be the binomial transform of a sequence (a_n), we can show that the boustrophedon transform of the sequence (t_n) of all 1's is simply the binomial transform of the Euler numbers. Let (t_n) be the sequence of all 1's and (s_n) its boustrophedon transform. Then we have that

$$s_n = \sum_{k=0}^{n} \binom{n}{k} t_k E_{n-k}.$$

Since $t_k = 1$ for all k, we have

$$\begin{aligned} s_n &= \sum_{k=0}^{n} \binom{n}{k} E_{n-k} \\ &= \sum_{k=0}^{n} \binom{n}{n-k} E_{n-k}. \end{aligned}$$

Setting $i = n - k$ yields $s_n = \sum_{i=0}^{n} \binom{n}{i} E_i$, which is the binomial transform of the Euler numbers.

3. The Path-Permutation Bijection Theorem

We seek to constructively establish a bijection between the set of paths from $(0,0)$ to (n,n) in the boustrophedon graph and $DU(n)$. This bijection relies on the following lemma.

Lemma 3.1. *Counting Principle: Suppose* $i \geq 1, j \geq 1, i + j \leq n$ *and* $x_1 < x_2 < x_3 < \cdots < x_{n-1} < x_n$. *Then* $x_i < x_{n-j+1}$.

Proof. Suppose to the contrary that there exists some $x_i \geq x_{n-j+1}$. Then $i \geq n - j + 1$ and $i + j \geq n + 1$, which is a contradiction. □

Theorem 3.2. *Let* $\mathcal{S}_n$ *be the set of all paths in the boustrophedon graph starting at* $(0,0)$ *and ending at* (n,n). *Then there exists a bijection*

$$\phi : \mathcal{S}_n \to DU(n).$$

Proof. ($\Rightarrow$) Consider a path in $\mathcal{S}_n$. For each $i, 1 \leq i \leq n$, define $f(i)$ to be the vertex where the path enters row i. Let x be the number of edges traversed by the path in row i. Then there is a downward arrow from $(i, f(i)+x_i)$ to $(i+1, f(i+1))$, and hence $f(i) + x_i + f(i+1) = i + 1$. Since $x_i \geq 0$,

$$f(i) + f(i+1) \leq i + 1 \tag{1}$$

We can define an element $\sigma = \sigma_1\sigma_2 \cdots \sigma_n \in DU(n)$ inductively as follows.

Let $\sigma_1 = n - f(n) + 1$ and $\sigma_2 = f(n-1)$. Then σ_1 is the $f(n)^{th}$ element of $1 < 2 < 3 < \cdots < n$ counting from the right and σ_2 is the $f(n-1)^{th}$ element counting from the left. By the counting principle, σ_1 and σ_2 are defined and $\sigma_2 < \sigma_1$. Suppose we have defined $\sigma_1\sigma_2\sigma_3 \cdots \sigma_{2j}$ so that $\sigma_1 \cdots \sigma_{2j}$ is down-up and let $y(1) < y(2) < \cdots < y(n - 2j + 1)$ be all the ordered elements of $[n] \setminus \{\sigma_1, \sigma_2, \cdots, \sigma_{2j-1}\}$. Then σ_{2j} is the $f(n-2j+1)^{\text{th}}$ element from the left, that is, $\sigma_{2j} = f(n-2j+1)$. Define σ_{2j+1} to be the $f(n-2j)^{\text{th}}$ element of $[n]\setminus\{\sigma_1, \sigma_2, \ldots, \sigma_{2j-1}\}$ from the right. So $\sigma_{2j+1} = y(n-2j+1-f(n-2j)+1)$. Since $f(n-2j) + f(n-2j+1) \leq n - 2j + 1$, we have $\sigma_{2j} < \sigma_{2j+1}$ by the counting principle.

Now let $z(1) < z(2) < z(3) < \cdots < z(n-2j)$ be all the ordered elements of $[n] \setminus \{\sigma_1, \cdots, \sigma_{2j}\}$. Since we have removed only σ_{2j} from the previous list, and $\sigma_{2j} < \sigma_{2j+1}$, it follows that σ_{2j+1} is still the $f(n-2j)^{\text{th}}$ element from the right. That is, $\sigma_{2j} = z(n - 2j - f(n-2j) + 1)$. Define σ_{2j+2} to be the $f(n-2j-1)^{\text{th}}$ element of $[n]\setminus\{\sigma_1, \ldots, \sigma_2 j\}$ from the left. So $\sigma_{2j+2} = f(n-2j-1)$, and since $f(n-2j-1) + f(n-2j) \leq n - 2j$, the counting principle implies $\sigma_{2j+2} < \sigma_{2j+1}$. Thus we have defined an element of $DU(n)$.

($\Leftarrow$) Given a permutation $\sigma = \sigma_1\sigma_2\cdots\sigma_n \in DU(n)$, define the set of pairs $\{(\ell, f(\ell))|1 \leq \ell \leq n\}$ by the equations

$$f(n-2j) = n+1-|\{\sigma_i : \sigma_i > \sigma_{2j+1}, i < 2j\}| - \sigma_{2j+1} \tag{2}$$

$$f(n-2j-1) = \sigma_{2j+2} - |\{\sigma_i : \sigma_i < \sigma_{2j+2}, i < 2j+1\}|, \tag{3}$$

where $0 \leq j < \frac{n+1}{2}$. This set of pairs uniquely determines a path in the boustrophedon graph by taking each pair $(\ell, f(\ell))$ to be the vertex at which it enters the ℓ^{th} row. To verify that this actually determines such a path, it suffices to show that $f(\ell) \neq 0$ for $1 \leq \ell \leq n$. So we have

$$f(n-2j) = n+1-|\{\sigma_i : \sigma_i > \sigma_{2j+1}, i < 2j\}| - \sigma_{2j+1},$$

and since $\sigma_{2j+1} = k$ for some $k \leq n$, it follows that

$$\begin{aligned} f(n-2j) &= n+1-|\{\sigma_i : \sigma_i > k, i < 2j\}| - k \\ &\geq n+1-(n-k)-k \\ &= 1. \end{aligned}$$

Similarly, we have

$$f(n-2j-1) = \sigma_{2j+2} - |\{\sigma_i : \sigma_i < \sigma_{2j+2}, i < 2j+1\}|,$$

and since $\sigma_{2j+2} = k$ for some $k \leq n$, it follows that

$$\begin{aligned} f(n-2j-1) &= k - |\{\sigma_i : \sigma_i < k, i < 2j+1\} \\ &\geq k-(k-1) \\ &= 1. \end{aligned}$$

By construction, applying the function ϕ to this path results in the given permutation σ. $\square$

Example 6. Consider the permutation $\sigma = 316274958$. We map σ to a set of vertices, fixing a path on the boustrophedon graph, using equations (2) and (3).From this we obtain $f(1) = 1, f(2) = 1, f(3) = 1, f(4) = 1, f(5) = 3, f(6) = 1, f(7) = 4, f(8) = 1, f(9) = 7$. Applying ϕ to this path, we generate the permutation with $\sigma_1 = 3, \sigma_2 = 1, \sigma_3 = 6, \sigma_4 = 2, \sigma_5 = 7, \sigma_6 = 4, \sigma_7 = 9, \sigma_8 = 5, \sigma_9 = 8$, giving us our starting permutation $\sigma = 316274958$.

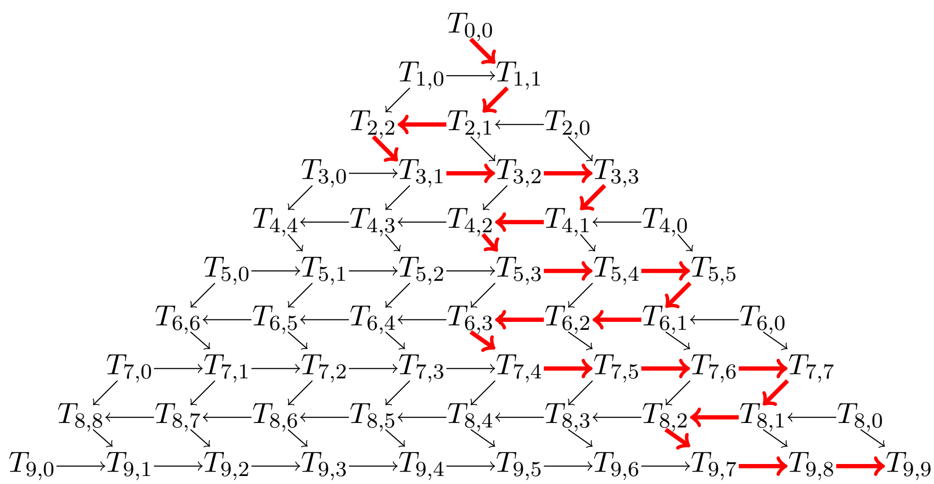


Figure 4. The path corresponding to Example 6

Example 7. Now consider $\sigma = 827361549$. From equations (2) and (3), we obtain $f(1) = 1, f(2) = 1, f(3) = 2, f(4) = 1, f(5) = 2, f(6) = 2, f(7) = 2, f(8) = 2, f(9) = 2$, which corresponds to the path shown in Figure 5. Then applying ϕ to this path, we generate the permutation given by $\sigma_1 = 8, \sigma_2 = 2, \sigma_3 = 7, \sigma_4 = 3, \sigma_5 = 6, \sigma_6 = 1, \sigma_7 = 5, \sigma_8 = 4, \sigma_9 = 9$. So $\sigma = 827361549$, which is our starting permutation.

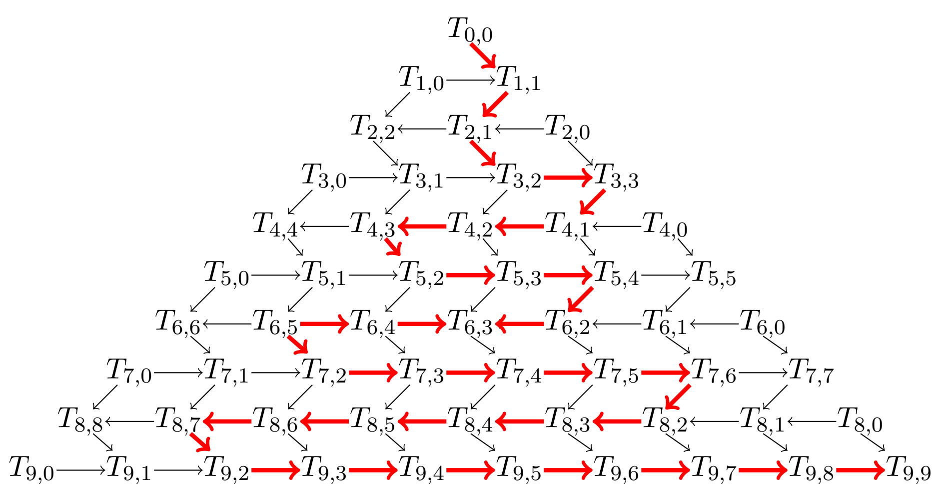


Figure 5. The path corresponding to Example 7

4. Umbral Rules

We now give a brief overview of umbral methods before applying them to the boustrophedon transform. Let (a_n) be a sequence of real numbers. The exponential generating function (EGF) of (a_n) is given by the formal power series

$$A(x) = \sum_{n=0}^{\infty} \frac{a_n}{n!} x^n.$$

Substituting a^n for a_n in the above equation, we obtain

$$\sum_{n=0}^{\infty} \frac{a_n}{n!} x^n = \sum_{n=0}^{\infty} \frac{a^n}{n!} x^n = e^{ax}.$$

This substitution gives a closed form for $A(x)$. The mapping $a_n \to a^n$ is known as the umbral substitution. We denote it by $a_n \to A^n$, where A is understood as an indeterminate called the umbra of (a_n) to reduce confusion.

We can view the umbral substitution as a linear functional. Let $\mathbb{R}[A]$ denote the ring of polynomials in the indeterminate A with real coefficients, understood as a vector space over $\mathbb{R}$. Define the linear functional

$$L : \mathbb{R}[A] \to \mathbb{R}$$

on the basis $\{A^n | n \geq 0\}$ of $\mathbb{R}[A]$ by

$$L(A^n) = a_n.$$

Given sequences (a_n) and (b_n), we define the umbral substitution $L : \mathbb{R}[A, B] \to \mathbb{R}$ on the basis $\{A^n B^m : n, m \geq 0\}$ by $L(A^n B^m) = a_n b_m$. We observe here that $L(A^n B^m) = L(A^n)L(B^m)$. We can define the umbral substitutions for any number of sequences in a similar way. From this definition, we obtain the following rules:

(1) If (a_n) is a sequence with EGF $A(x)$, then $A(x) = L(e^{Ax})$.

(2) $\frac{d^k}{dx^k} A(x) = L(A^k e^{Ax})$.

(3) If $C(x) = A(x)B(x)$, where $A(x)$, $B(x)$, and $C(x)$ are the EGFs of the sequences (a_n), (b_n), and (c_n) respectively, then $c_n = L((A+B)^n)$.

The first two rules require that we extend L to the space $\mathbb{R}[[A]]$ of formal power series.

Let (E_n) be the sequence of Euler numbers. Theorem 3.2 implies that the number of down-up permutations on $[n]$ is equal to the number of paths from $(0, 0)$ to (n, n) in the boustrophedon graph. From this, the following equation for the boustrophedon transform can be derived:

$$b_n = \sum_{k=0}^{n} \binom{n}{k} a_k E_{n-k}. \tag{4}$$

We now introduce a general transformation before proving results of the boustrophedon transform. Let (a_n) and (c_n) be fixed sequences of real numbers and define a new sequence (s_n) by the transformation.

$$s_n = \sum_{k=0}^{n} \binom{n}{k} a_k c_{n-k}. \tag{5}$$

We now prove a general result relating the umbrae of these sequences.

Proposition 4.1. *Let A, C, and S be umbrae corresponding to (a_n), (c_n), and (s_n), respectively. Then $L(A^n) = L((S - C)^n)$ for all $n \geq 1$.*

Proof.

$$\begin{aligned} L(S^n) = s_n &= \sum_{k=0}^{n} \binom{n}{k} a_k c_{n-k} \\ &= \sum_{k=0}^{n} \binom{n}{k} L(A^k) L(C^{n-k}) \\ &= \sum_{k=0}^{n} \binom{n}{k} L(A^k C^{n-k}) \\ &= L\left(\sum_{k=0}^{n} \binom{n}{k} A^k C^{n-k}\right). \end{aligned}$$

From the binomial theorem (see Proposition 6.1 in the Appendix), it follows that

$$L(S^n) = L((A + C)^n).$$

Since $\{S^n | n \geq 0\}$ is a basis for $\mathbb{R}[S]$, it follows that $L(p(S)) = L(p(A + C))$ because if $p(S) = \sum_{k=0}^{d} \alpha_k S^k$, then

$$\begin{aligned} L(p(S)) &= L\left(\sum_{k=0}^{d} \alpha_k S^k\right) \\ &= \sum_{k=0}^{d} \alpha_k L(S^k) \\ &= \sum_{k=0}^{d} \alpha_k L((A + C)^k) \\ &= L\left(\sum_{k=0}^{d} \alpha_k (A + C)^k\right) \\ &= L(p(A + C)). \end{aligned}$$

So consider the polynomial $p(S) = (S - C)^n$. Then $L(p(S)) = L(p(A + C))$ implies that $L((S - C)^n) = L((A + C - C)^n) = L(A^n)$. Thus we obtain the relation $L(A^n) = L((S - C)^n)$. □

We now derive the inverse of this transformation.

Proposition 4.2. *The inverse of the transformation (5) is given by the equation*

$$a_n = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} s_k c_{n-k}.$$

Proof. By Proposition 4.1, we have

$$\begin{aligned} a_n &= L(A^n) \\ &= L((S - C)^n) \\ &= L\left(\sum_{k=0}^{n} \binom{n}{k} S^k (-C)^{n-k}\right) \\ &= \sum_{k=0}^{n} \binom{n}{k} L(S^k) L((-C)^{n-k}) \\ &= \sum_{k=0}^{n} \binom{n}{k} L(S^k) L((-1)^{n-k} C^{n-k}) \\ &= \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} L(S^k) L(C^{n-k}) \\ &= \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} s_k c_{n-k} \end{aligned}$$

□

We obtain the following corollary for the inverse boustrophedon transform by taking the sequence (c_n) to be the Euler numbers (E_n).

Corollary 4.3. *The inverse of the boustrophedon transform (4) is given by the equation*

$$a_n = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} b_k E_{n-k}$$

for $n \geq 1$.

Proposition 4.4. *The EGF of the sequence of Euler numbers* (E_n) *is given by* $E(x) = \tan x + \sec x$.

Proof. Let E and F be independent umbrae of (E_n). Then $L(E^n) = E_n$ and $L(F^n) = E_n$. Let $f(x)$ denote the EGF of (E_n). Then

$$f(x) = L(e^{Ex}) = L(e^{Fx}).$$

It can be shown that $2E_{n+1} = \sum_{k=0}^{n} \binom{n}{k} E_k E_{n-k}$, which implies that

$$\begin{aligned} 2L(E^{n+1}) &= \sum_{k=0}^{n} \binom{n}{k} L(E^k) L(F^{n-k}) \\ &= L\left(\sum_{k=0}^{n} \binom{n}{k} E^k F^{n-k} \right) \\ &= L((E+F)^n) \end{aligned}$$

for $n \geq 1$. Thus, $2L(E^{n+1}) = L((E+F)^n)$ for $n \geq 1$. Multiplying both sides by $\frac{x^n}{n!}$ and summing over $n \in \mathbb{N}$, we obtain

$$2L\left(\sum_{n=1}^{\infty} \frac{E^{n+1}x^n}{n!} \right) = L\left(\sum_{n=1}^{\infty} \frac{(E+F)^n x^n}{n!} \right).$$

Equivalently,

$$2L\left(\sum_{n=0}^{\infty} \frac{E^{n+1}x^n}{n!} - E \right) = L\left(\sum_{n=0}^{\infty} \frac{((E+F)x)^n}{n!} - 1 \right),$$

but

$$\begin{aligned} 2L\left(\sum_{n=0}^{\infty} \frac{E^{n+1}x^n}{n!} - E \right) &= 2L\left(\sum_{n=0}^{\infty} \frac{E^{n+1}x^n}{n!} \right) - 2L(E) \\ &= 2L\left(E \sum_{n-0}^{\infty} \frac{E^n x^n}{n!} \right) - 2E_1 \\ &= 2L(Ee^{Ex}) - 2 \\ &= 2f'(x) - 2, \end{aligned}$$

and

$$\begin{aligned} L\left(\sum_{n=0}^{\infty} \frac{(E+F)^n x^n}{n!} - 1 \right) &= L\left(\sum_{n=0}^{\infty} \frac{(E+F)^n x^n}{n!} \right) - L(1) \\ &= L(e^{(E+F)x}) - 1 \\ &= L(e^{Ex}e^{Fx}) - 1 \\ &= L(e^{Ex})L(e^{Fx}) - 1 \\ &= f(x)f(x) - 1 \\ &= (f(x))^2 - 1. \end{aligned}$$

So $2f'(x) - 2 = (f(x))^2 - 1$. Rearranging the terms, we obtain the initial value problem given by

$$\begin{cases} 2f'(x) = (f(x))^2 + 1 \\ f(0) = 1 \end{cases}$$

which has solution $f(x) = \tan\left(\frac{x}{2} + \frac{\pi}{4}\right) = \tan x = \sec x$ (see Propositions 6.2 and 6.3 in the Appendix). Therefore, $E(x) = \tan x + \sec x$. □

Corollary 4.5. *Let (a_n) be a sequence with boustrophedon transform (b_n). Let $A(x)$ and $B(x)$ be their respective EGFs. Then $B(x) = A(x)(\tan x + \sec x)$.*

Proof.

$$\begin{aligned} B(x) = L(e^{Bx}) &= L\left(\sum_{n=0}^{\infty} \frac{B^n}{n!} x^n\right) \\ &= \sum_{n=0}^{\infty} \frac{L(B^n)}{n!} x^n \\ &= \sum_{n=1}^{\infty} \frac{L((A+E)^n x^n)}{n!} \\ &= L\left(\sum_{n=0}^{\infty} \frac{(A+E)^n}{n!} x^n\right) \\ &= L(e^{Ax})L(e^{Ex}) \\ &= A(x)(\tan x + \sec x). \end{aligned}$$

□

5. Conclusion

The properties of the boustrophedon transform are readily explored through umbral methods. An exploration of the transform's properties yields the establishment of a bijection between the paths of the boustrophedon triangle and the set of alternating permutations on $[n]$. Further research remains to be done on transforms similar to the boustrophedon transform and on the boustrophedon transform of sequences not explored here (although most sequences in the OEIS.org database were addressed during our research).

The first 200 terms of the boustrophedon transform of the sequence defined by $a_k = 2E_{k+1} + E_k$, where E_k is the kth Euler up/down number, sequence A104854 in the OEIS.org database, match the sequence defined as the number of permutations of $[n]$ with fewer than two interior elements having values lying between the values of their neighbors, sequence A226435 in the database. This implies that the sequences might be the same, but further research into sequence A226435 is needed before a proof can be attempted.

Furthermore, there may be other combinations of the sequence of Euler up/down numbers that, when the boustrophedon transform is taken, result in combinatorially significant results. Specifically, more research is needed to determine if a general solution to the set of sequences $t(n,k)$ defined as the number of permutations of $[n]$ with fewer than k interior elements having values lying between the values of their neighbors can be expressed using the boustrophedon transform of a combination of the Euler up/down numbers.

6. Appendix

Proposition 6.1. $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$.

Proof.

$$\begin{aligned}(x+y)^n &= \left(y\left(\frac{x}{y}+1\right)\right)^n \\ &= y^n \sum_{k=0}^{n} \binom{n}{k} \left(\frac{x}{y}\right)^k \\ &= \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} \quad \square\end{aligned}$$

Proposition 6.2. *The initial value problem*

$$\begin{cases} 2f'(x) = (f(x))^2 + 1 \\ f(0) = 1 \end{cases}$$

has solution $f(x) = \tan\left(\frac{x}{2} + \frac{\pi}{4}\right)$.

Proof. Setting $f(x) = y$, we get $2\frac{dy}{dx} = y^2 + 1$. So $\frac{dy}{y^2+1} = \frac{dx}{2}$. Integrating both sides gives $\arctan(y) = \frac{x}{2} + C$. So $y = \tan(\frac{x}{2} + C)$. Since $f(0) = 1$, we have $1 = \tan(C)$ which implies $C = \frac{\pi}{4}$. Thus $y = \tan(\frac{x}{2} + \frac{\pi}{4})$. $\square$

Proposition 6.3. $\tan\left(\frac{x}{2} + \frac{\pi}{4}\right) = \tan x + \sec x$.

Proof. By the angle sum formula for the tangent function,

$$\begin{aligned}\tan\left(\frac{x}{2} + \frac{\pi}{4}\right) &= \frac{\tan(\frac{x}{2}) + \tan(\frac{\pi}{4})}{1 - \tan(\frac{x}{2})\tan(\frac{\pi}{4})} \\ &= \frac{\tan(\frac{x}{2}) + 1}{1 - \tan(\frac{x}{2})}.\end{aligned}$$

Using the double-angle formula for the tangent function, we get

$$\tan x = \frac{2\tan(\frac{x}{2})}{1 - \tan^2(\frac{x}{2})}.$$

Now let $t = \tan(\frac{x}{2})$. Through substitution it follows that $\tan x = \frac{2t}{1-t^2}$ and $\tan\left(\frac{x}{2}+\frac{\pi}{4}\right) = \frac{t+1}{1-t}$. Using the Pythagorean theorem, we get

$$(2t)^2 + (1-t^2)^2 = (t^2+1)^2.$$

So we have $\tan x = \frac{2t}{1-t^2}$ and $\sec x = \frac{t^2+1}{1-t^2}$. Then it follows that

$$\begin{aligned}\tan x + \sec x &= \frac{t^2+1}{1-t^2} + \frac{2t}{1-t^2}\\ &= \frac{(t+1)^2}{1-t^2}\\ &= \frac{t+1}{1-t}\\ &= \tan\left(\frac{x}{2}+\frac{\pi}{4}\right).\end{aligned}$$

Therefore, $\tan\left(\frac{x}{2}+\frac{\pi}{4}\right) = \tan x + \sec x$. □

6.1. Mathematica Code for searching the OEIS.org database.

```
(* The OEIScrape module has been adapted from code
freely available at:
http : //www.brotherstechnology.com/objects/OEIS_Automated_Seach.nb*)

(* Instructions for acquiring seqtranslib.m available at
http : //oeis.org/seqtranslib.html *)
<< seqtranslib.m;

(* stripped.txt available at http : //oeis.org/stripped.gz*)
SequenceList = Import["stripped", "CSV"];

OEIScrape[seq_List]:=
Module[{urlSuffix, data, url, sorry, strike, rfound, results,
pages, resultPage, p, bingo}, {

urlSuffix = {};
Do[urlSuffix = urlSuffix <> ToString@seq[[k]] <> "%2C%20",
{k, 1, Length@seq}];
data = Import[url = "http://oeis.org/search?q=" <> urlSuffix];

Do[{
sorry = 0;

strike = StringCases[data, "Sorry"]; sorry = Length@strike;
If[sorry > 0, {
If[VerboseAll==True, Print["Sequence: ", SequenceList[[n]][[1]],
"Found no matches."], Null], Break[]}, Null];
```

```
If[Length@StringCases[data, DigitCharacter.. ~~ _ ~~ "result found"] == 0,
rfound = StringCases[data, DigitCharacter.. ~~ _ ~~ "results found"],
rfound = StringCases[data, DigitCharacter.. ~~ _ ~~ "result found"]];
results = ToExpression@StringCases[rfound, (DigitCharacter..)][[1, 1]];

If[Or[VerboseAll == True, VerboseResults = True],
Print["Sequence: ", SequenceList[[n]][[1]], " Found ",
results, " matches "], Null];

AppendTo[OutputList, {n, SequenceList[[n]][[1]], results, PostSeq}]
}, {1}]
}]
Off[General::partw];
Off[FetchURL::httperr];
Off[StringCases::strse];

FirstSequence = 1;
LastSequence = 200;

(*Setthelevelofoutput : VerboseAllreturnstheresultof
everysequence, VerboseResultsreturnsonlythematched
sequences(onlyifVerboseAllisFalse)*)
VerboseAll = False;
VerboseResults = True;

OutputFileName = "Matchlist.txt"

OutputList = {};

MemoryConstrained[
Do[{
PreSeq = Range[15];
Do[PreSeq[[i − 1]] = SequenceList[[n]][[i]], {i, 2, 16}];
OEIScrape[PostSeq = BoustrophedonBisTransform[PreSeq]];
}, {n, FirstSequence, LastSequence}],
500434656]
Print["Sequence search complete."]
Export[OutputFileName, OutputList, "CSV"];
```

6.2. Mathematica code that explores patterns in the sequences.

```
(* This code uses the OEIS seqtranslib library
found at http : //oeis.org/seqtranslib.html*)
```

```
(* Import the sequence transfrom library *)
<< seqtranslib.m

(* Define an input sequence *)
(* this can be a sequence in generic or specified form *)
inputTable = Table[1, {n, 10}];
inputSeq = {inputTable/.List → Sequence};

(* Apply the transform *)
outputSeq = BoustrophedonBisTransform[inputSeq];
Print["Input Sequence:", inputSeq];
Print["Transformed Sequence: ", outputSeq];
primeTable = {2, 3, 5, 7, 11};
p[x_]:=primeTable[[x]];

(*(Checkfor mod patternswherekisthedivisor()
6 = (0 mod 3), 3 ∼= k)*)
(* added stuff to check mod powers of primes *)
(* the formating could be better *)
triangularArrayLayout[triArray_List, opts___]:=
Module[{n = Length[triArray]}, Graphics[MapIndexed[Text
[Style[#1, Large], {Sqrt[3](n − 1 + #2.{−1, 2}),
3(n − First[#2] + 1)}/2]&, triArray, {2}], opts]];

triangleSeq = {{inputSeq[[1]]}, {inputSeq[[2]],
outputSeq[[2]]}, {outputSeq[[3]], inputSeq[[3]]
+outputSeq[[2]], inputSeq[[3]]}, {inputSeq[[4]],
inputSeq[[4]] + outputSeq[[3]], inputSeq[[4]]+
outputSeq[[3]] + inputSeq[[3]] + outputSeq[[2]],
outputSeq[[4]]}, {outputSeq[[5]], inputSeq[[5]]
+outputSeq[[4]] + inputSeq[[4]] + outputSeq[[3]]
+inputSeq[[3]] + outputSeq[[2]] + inputSeq[[4]]+
outputSeq[[3]], inputSeq[[5]] + outputSeq[[4]]+
inputSeq[[4]] + outputSeq[[3]] + inputSeq[[3]]+
outputSeq[[2]], inputSeq[[5]] + outputSeq[[4]],
inputSeq[[5]]}}

triangularArrayLayout[triangleSeq]


"pre transform"
premodTransformTable = Column[Table[{"\n", p[x], k,
Mod[inputSeq, p[x]^k]}, {x, 1, 5}, {k, 1, 10}]]
"post transform"
postmodTransformTable = Column[Table[{"\n", p[x],
```

```
k, Mod[outputSeq, p[x]^k]}, {x, 1, 5}, {k, 1, 10}]]

"pre Transform"
premodTransformTable = Column[Table[{k, Mod[inputSeq, k]},
{k, 2, 10}]]
"post Transform"
postmodTransformTable = Column[Table[{k, Mod[outputSeq, k]},
{k, 2, 10}]]

(*prettyselfexplanatory, messwiththesequence*)
modifiedInput = inputSeq + outputSeq
modifiedOutput = outputSeq − inputSeq

(*runbackthroughttheprevioustwothingsbutwithm
odifiedsequences*)
"pre transform"
premodTransformTable = Column[Table[{"\n", p[x], k,
Mod[modifiedInput, p[x]^k]}, {x, 1, 5}, {k, 1, 10}]]
"post transform"
postmodTransformTable = Column[Table[{"\n", p[x], k,
Mod[modifiedOutput, p[x]^k]}, {x, 1, 5}, {k, 1, 10}]]

"pre Transform"
premodTransformTable = Column[Table[{k, Mod[modifiedInput,
k]}, {k, 2, 10}]]
"post Transform"
postmodTransformTable = Column[Table[{k, Mod[modifiedOutput,
k]}, {k, 2, 10}]]
```

References

[1] Miller, J. & Sloane, N.J.A. & Young, N.E., *A New Operation on Sequences: The boustrophedon Transform**, J. Combinatorial Theory, Series A, volume 76, pages 44-54, 1996.

[2] Roman, S. & Rota, G., *The Umbral Calculus*, Volume 27, Number 2, pages 95-120. New York & London: Academic Press, 1978.

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