Efficient packings of unit squares in a large square

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Abstract

How efficiently can a large square of side length x be packed with non-overlapping unit squares? In this note, we show that the uncovered area $W(x)$ can be made as small as $O(x^{3/5})$. This improves an earlier estimate which showed that $W(x) = O(x^{\frac{3+\sqrt{2}}{7}} \log x)$.

1 Introduction

Given a large square $S(x)$ of side length x, let $W(x)$ denoted the minimum possible amount of uncovered area in any packing of $S(x)$ with non-overlapping unit squares. When $x = N$ is an integer, then of course $W(N) = 0$. However, when x is not an integer, then some waste (= uncovered area) is inevitable. In we naively pack the square in the "obvious" way, (see Fig. 1), then the wasted area will be proportional to $x(x - \lfloor x \rfloor)$, which is *linear* in x when $(x - \lfloor x \rfloor)$ is bounded away from 0. During the past 40 years, better upper bounds for $W(x)$ have been obtained. Beginning with the estimate $W(x) = O(x^{\frac{7}{11}}) = O(x^{0.63636\ldots})$ from [3], this was improved by Montgomery [7] to $W(x) = O(x^{\frac{3-\sqrt{3}}{2}}) = O(x^{0.63397\ldots})$ and then most recently to $W(x) = O(x^{\frac{3+\sqrt{2}}{7}} \log x)$ by the authors [2] where $\frac{3+\sqrt{2}}{7} = 0.63060\ldots$.

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In the other direction, a seminal result of Roth and Vaughan [8] shows that if $x(x - \lfloor x \rfloor) > \frac{1}{6}$ then

$$W(x) > 10^{-100}\sqrt{x\ \|x\|} \tag{1}$$

where $\|x\|$ denote the distance from x to the nearest integer. In particular, this shows that $W(x) > cx^{1/2}$ for some absolute constant c provided x is bounded away from an integer.

In this note, we improve the upper bound for $W(x)$.

Theorem 1.

$$W(x) = O(x^{\frac{3}{5}}).$$

To prove Theorem 1, we will describe the packing procedure in three stages. In the first stage, we partition the square $S(x)$ into a perfect square (of integral side) and two rectangles. In second stage, the rectangles are packed by stacks of unit squares but some trapezoids are still left uncovered as well as some tiny triangles along the edges. At the last stage, we pack the trapezoids in a careful way. We note that in the first and second stages, the strategy for packing $S(x)$ will be similar to that used in [2], although the parameters are chosen differently.

In each stage, there are a number of steps to pack unit squares into the regions. During this process, we will keep track of the waste in each step of the way. All together, we will consider nine types of waste, denoted by $W_i(x)$ for $i = 1, \ldots, 9$. We will show that for each i, $W_i(x) = O(x^{3/5})$. Summing the $W_i(x)$ for $i = 1, \ldots, 9$, then yields the claimed upper bound for $W(x)$.

2 Packing rectangles with unit squares.

Since we are only proving asymptotic bounds on $W(x)$, we will omit floor and ceiling functions when they are not essential. In order to simplify the notation, we are first going to assume that $x = N + 1/2$ for some integer N. The modifications needed for general x should be clear.

For the first stage of the construction, we will pack most of $S(x)$ in the naive way but leaving two unpacked rectangles with short side lengths equal to $x^{4/5}$ (see Fig. 1), In particular, we can take this length to be of the form $M+1/2$ for some integer M which makes $M+1/2$ closest to $x^{4/5}$. Note that the figures are not drawn to scale.

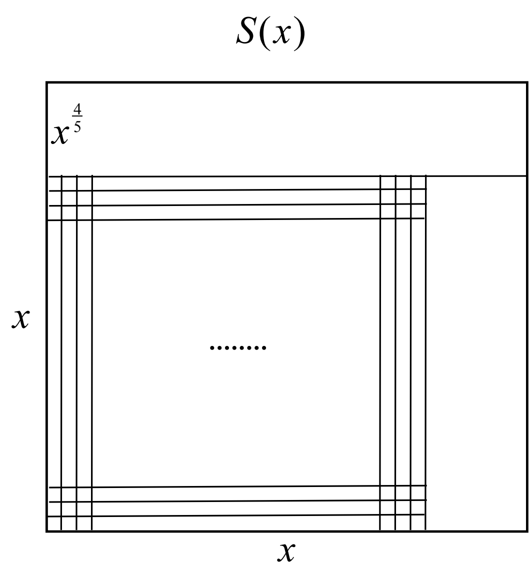


Figure 1: *Packing most of $S(x)$ with axis-aligned stacks of unit squares .*

Next we are going to take stacks of $M+1$ unit squares, tilt them slightly and pack most of the two rectangles with these stacks.

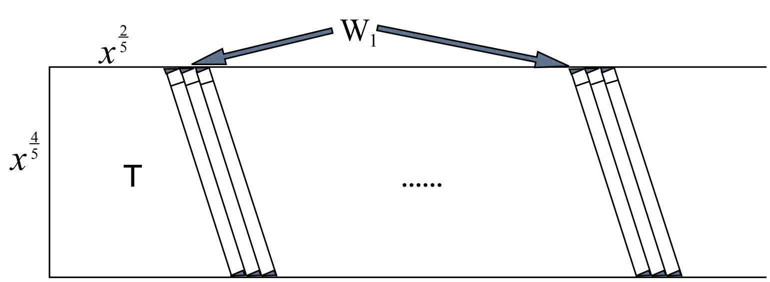
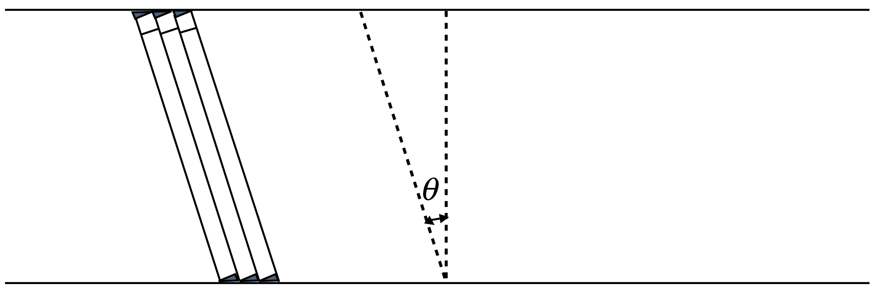


Figure 2: *Packing the rectangles with stacks of squares.*

There are now two questions. How much of the rectangles do we pack this way, and how much waste do we generate?



$$\theta = \arctan\frac{\sqrt{(M+1+\tan\theta)^2-(M+\frac{1}{2})^2}}{M+\frac{1}{2}} \approx \frac{1}{\sqrt{M}} \approx \frac{1}{x^{\frac{2}{5}}}+O(\frac{1}{x^{\frac{4}{5}}})$$

Figure 3: *The angle of a stack.*

Referring to Fig. 3, a computation shows that the angle θ in the small right triangle is asymptotic to $\frac{1}{\sqrt{M}}$ (since $\tan\theta \sim \theta$ as $M \to \infty$.) Hence, the area of each of the shaded right triangle is (asymptotic to) $\frac{1}{\sqrt{M}} = O(x^{-2/5})$. Since there are at most $4x$ such triangles, then this waste $W_1(x)$ is bounded by $W_1(x) = O(x^{3/5})$.

3 Packing the trapezoids with unit squares.

We will fill the rectangles with the aforementioned stacks until we have empty trapezoids with top edges of length $x^{2/5}$. Hence, the bottom sides have length $2x^{2/5}$ (see Fig. 4).

We will focus on a single trapezoid T (we treat the other three in the same way). We subdivide T into $t+1$ trapezoids B_i, $0 \le i \le t$, as follows. Let b_0 denote the length of the top edge of T. Then go down the left-hand side of T by steps of size 1 until the horizontal line going to the right-hand side of T has length b_1 exceeding $b_0 + 1$. From the bound for θ this length is at most

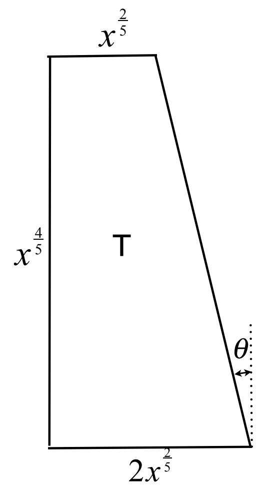


Figure 4: *The trapezoid T.*

$b_0+1+O(x^{-2/5})$. This will happen when the horizontal line has come down for a distance of $h_0 \sim x^{2/5}$ where h_0 is an integer. We now draw this horizontal line which will form the bottom edge of the first subtrapezoid B_0. Now we repeat this process starting from the bottom edge of B_0. Namely, we go down until we find that the horizontal line to the right-hand side of T has length b_2 which exceeds b_1+2 but is less than $b_1+2+O(x^{-2/5})$. Again, this will happen when the left-hand side has come down a distance $h_1 \sim x^{2/5}$. Drawing the horizontal line at this point will form the bottom edge of the subtrapezoid B_1. We keep doing this until we reach the bottom of T. In general, B_i will have a top edge of length b_i between b_0+i and $b_0+i+O(x^{-2/5})$, a left-hand edge of *integer* length $h_i \sim x^{2/5}$ and a bottom edge of length b_{i+1} between b_0+i+1 and $b_0+i+1+O(x^{-2/5})$ (see Fig. 5). The exception occurs with the last trapezoid B_t. For this subtrapezoid, $h_t = O(x^{2/5})$ may not be an integer. In this case, we just pack B_t "trivially" with axis-aligned stacks of squares, leaving a waste altogether of $W_2(x) = O(x^{2/5})$.

In the next section, we describe how to pack squares into the B_i.

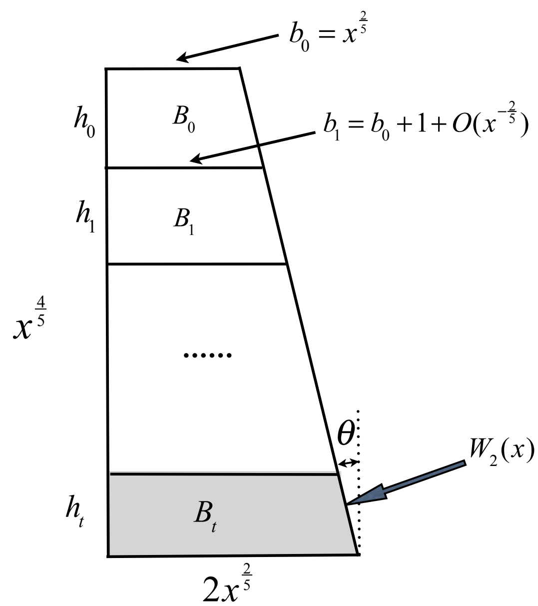


Figure 5: *Subividing T into subtrapezoids B_i.*

4 Packing the B_i.

The first step in packing B_i is to place i stacks of unit squares of height h_i along the left-hand edge of B_i (see Fig. 6).

Since h_i is an integer, no wasted space is created here. By construction, the remaining portion of B_i is a subtrapezoid essentially isomorphic to B_0, having a top edge length of $b_i - i = b_0 + O(\frac{1}{x^{2/5}})$.

The next step will be to remove $x^{1/5}$ small right triangles from the right-hand edge of B_i as shown in Fig. 7.

Each triangle has height $x^{1/5}$. The area of such a triangle is $O((x^{1/5})^2\theta) = O(1)$. Hence, the total area of all these triangles is $O(x^{3/5})$ since there are $x^{2/5}$ B_i's and each B_i has $O(x^{1/5})$ such triangles. These triangles will be left empty so the waste introduced here is $W_3(x) = O(x^{3/5})$.

Now define m to be $\lceil b_0 \rceil + 1$. All of the remaining square packing of the B_i will be done with stacks of unit squares of height m. In particular, all of

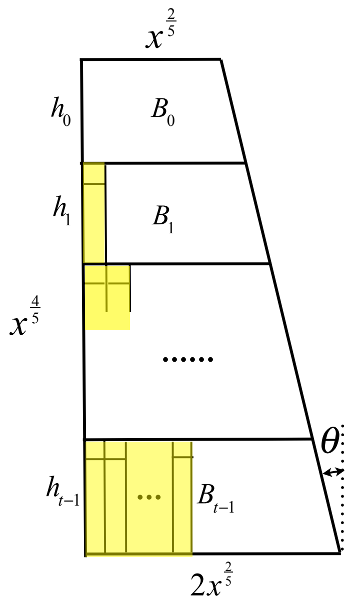


Figure 6: *Filling the ends of the B_i with stacks of squares.*

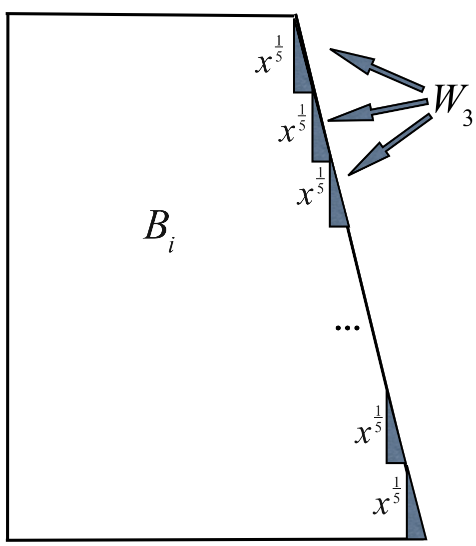


Figure 7: *Creating "serrated" B_i's.*

the stacks will have their left-hand ends touching the left-hand edges of the modified B_i (see Fig. 8).

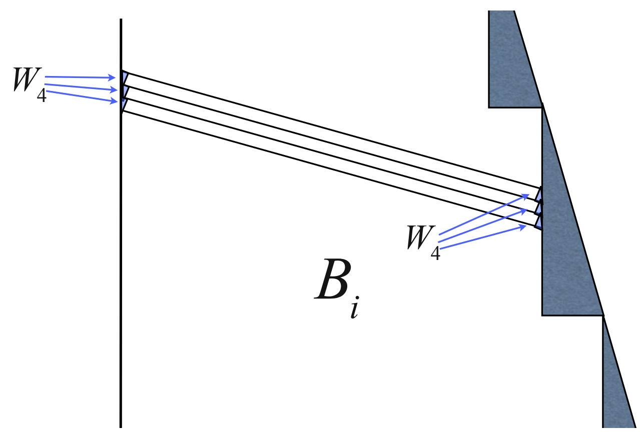


Figure 8: *Filling B_i with tilted stacks of height m.*

Also, these stacks will always be placed as close to the top as possible, so that each stack will touch the stack above it. We continue placing stacks of height m in the B_i, allowing stacks to cross from B_i to B_{i+1} when necessary. Doing this will create a certain amount of wasted space, which we now estimate.

First, the waste $W_4(x)$ created by the small triangles at the ends of the stack (see Fig. 8).

The "tilt" of any stack is at most $O(x^{-1/5})$ so that the area of such a triangle is $O(x^{-1/5})$. Since there are $O(x^{4/5})$ such triangles, then we see that $W_4(x) = O(x^{3/5})$.

Next, we examine what happens when consecutive stacks touch different parts of the "serrated" B_i (see Fig. 9).

A simple calculation shows that the difference in the tilts of the two stacks can be estimated as follows. The tilt of the upper stack is $\sim \frac{c}{x^{1/5}} + O(x^{-2/5})$ for some constant c between $\sqrt{2}$ and 2 (which depends on the fractional part of b_0). The tilt of the lower stack is $\sim \frac{c}{\sqrt{x^{2/5}+x^{-1/5}}}$. Thus, by ignoring the

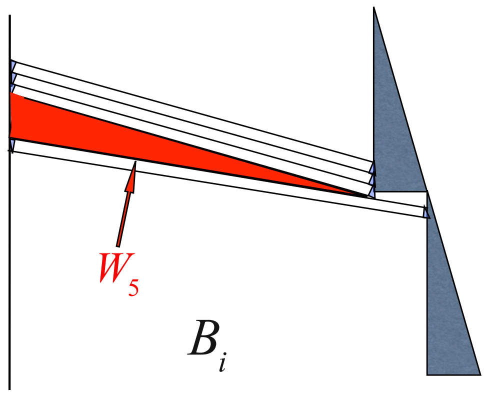


Figure 9: *A wasted triangle between two stacks within* B_i.

lower-order terms, the difference of the two tilts is

$$\begin{aligned} &\sim \frac{c}{x^{1/5}} - \frac{c}{\sqrt{x^{2/5} + x^{-1/5}}} = \\ &= \frac{c}{x^{1/5}} \left(1 - \frac{1}{\sqrt{1 + x^{-3/5}}}\right) = O(x^{-4/5}) \end{aligned}$$

so that the area of the elongated triangle between the two stacks is

$$O\big(x^{2/5}\big)^2 x^{-4/5} = O(1).$$

The same bound holds for the area of the elongated triangle between any two consecutive stacks toughing the serrated boundary.

Since there are at most $O(x^{3/5})$ such transitions inside the B_i's over all i, then this waste $W_5(x)$ is bounded by $W_5(x) = O(x^{3/5})$.

In addition, there is a small unpacked region above the lower of the two stacks which has area $O(x^{-(1/5)})$ (see Fig. 10).
These altogether contribute a waste of at most $W_6(x) = O(x^{2/5})$.

Finally, we have to consider what happens when the stacks cross the boundaries between consecutive B_i's (see Fig. 11).

As we can see in Fig. 11, there is an elongated triangle between the last stack (of length m) in B_i and the first stack (also of length m) in B_{i+1}. We

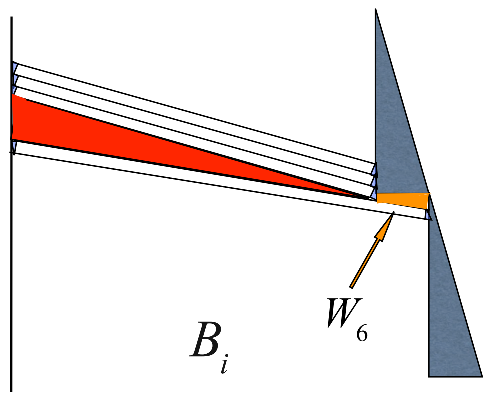


Figure 10: *Additional wasted space between stacks with slightly different tilts.*

note that the tilt of the first stack in B_i and the tilt of the first stack in B_{i+1} are asymptotically the same since $b_{i+1} - 1 = b_i + O(x^{-2/5}) \sim x^{2/5}$. We have shown previously that within B_i, there are $x^{1/5}$ gaps each of which has an angle of order $O(x^{-4/5})$. Therefore the difference of the tilts of the first stack and the last stack in B_i is at most $O(x^{-4/5})$. This implies the difference in the tilts between the the last stack in B_i and the first stack in B_{i+1} is $O(x^{-3/5})$. Thus the area of the region between the two stacks is

$$O((x^{2/5})^2 x^{-3/5}) = O(x^{1/5}).$$

Since there are just $O(x^{2/5})$ such transitions between the B_i's, then the waste $W_7(x)$ here is just $W_7(x) = O(x^{3/5})$.

There is still a small uncovered area of area at most 1 to the left of the first stack of B_{i+1} which we call waste W_8 (as seen in Fig. 12). Altogether, there are $x^{2/5}$ such transitions, so $W_8(x) = O(x^{2/5})$. There is also an uncovered area above the first stack in B_0 as well as an uncovered area below the last stack in B_{t-1}. Both can be packed trivially with a waste at most $W_9(x) = O(x^{2/5})$.

Adding up all the $W_i(x)$, for $i = 1, \ldots, 9$, we see that the wasted space in our packing is $O(x^{3/5})$ as claimed. This completes the proof of Theorem 1.

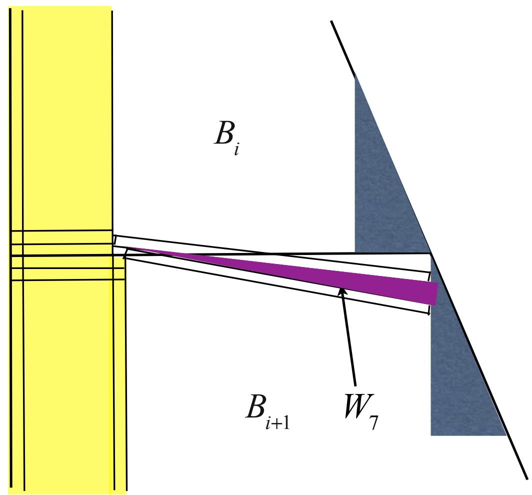


Figure 11: *Space wasted when the stacks end up in different B_i's.*

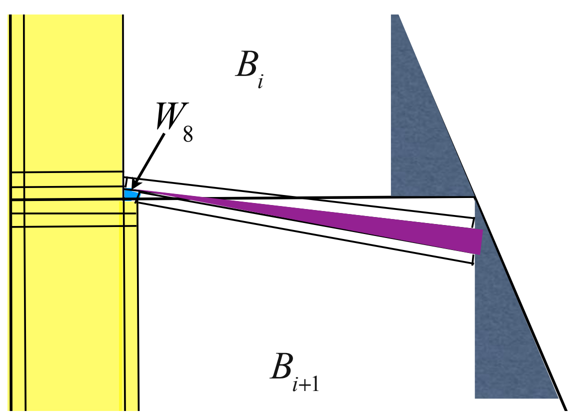


Figure 12: *Small uncovered area to the left of the first stack of B_{i+1}..*

5 Concluding remarks.

Of course, the basic problem is to determine the correct order of growth of $W(x)$. On one hand, because of the result of Roth and Vaughan [8], one might be tempted to guess that $W(x) = O(x^{1/2})$. (The authors don't believe this.) On the other hand, the best upper bound currently available is $W(x) = O(x^{3/5})$.

Challenge 1. ($ 100), Show that $W(x) = o(x^{3/5})$.

Challenge 2. ($ 250). Show that $W(x) = O(x^{3/5-c})$ for some fixed $c > 0$.

Challenge 3. ($ 500). Show that $W(N+1/2) \gg N^{1/2+c}$ for some fixed $c > 0$.

Of course, the rewards are also given for *disproving* the above assertions (but only to the first claimants).

We observe that the techniques used in the preceding results can be applied with minor changes to the question of *covering* a square of side x with a minimum number of unit squares (as was done on [2]). This results in the following.

Theorem 2. *It is possible to cover a square of side x with $x^2 + C(x)$ non-overlapping unit squares where $C(x) = O(x^{3/5})$.*

Earlier results on this variation also appear in [5, 6, 9].

We point out that there has been a fair amount of work on optimally packing a small number (e.g., less than 100) of unit squares into the smallest possible square. A good survey of these results can be found in [1] and [4].

6 Acknowledgement

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