On the Bott-Chern characteristic classes for coherent sheaves

Hua Qiang

Abstract

In the paper [Blo10], Block constructed a dg-category $\mathcal{P}_{\mathcal{A}^{0,\bullet}(X)}$ using cohesive modules which is a dg-enhancement of $D^b_{\mathrm{Coh}}(X)$, the bounded derived category of complexes of analytic sheaves with coherent cohomology. In this article, we construct natural superconnections on cohesive modules and use them to define characteristic classes with values in Bott-Chern cohomology. In addition, we generalize the double transgression formulas in [BGS88a] [BC65] [Don87] and prove the invariance of these characteristic classes under derived equivalences. This provides an extension of Bott-Chern characteristic classes to coherent sheaves on complex manifolds.

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1   Introduction

Traditionally, the complex structure of a complex manifold X is encoded in the sheaf of holomorphic functions $\mathcal{O}_X$. For applications to noncommutative geometry, such local constructions are not available and we are forced to use global differential geometric constructions. When the manifold is projective, every coherent sheaf $\mathscr{S}$ admits a global resolution by holomorphic vector bundles

$$0 \to E^n \to E^{n-1} \to ... \to E^1 \to E^0 \to \mathscr{S} \to 0$$

and we can apply the theory of holomorphic vector bundles to study $\mathscr{S}$. However, for a general compact complex manifold that is not projective, there may not exist a global resolution by holomorphic vector bundles and alternative methods are required.

Even though the sheaf of holomorphic sections of $\mathscr{S}$ does not admit global resolution by holomorphic vector bundles, the underlying sheaf of real analytic sections does admit a global resolution by real analytic vector bundles, see [AF61]. In [Blo10][Blo06], Block constructed higher order differentials on such resolutions and the resulting geometric objects are called cohesive modules. He further showed that they form a dg-category $\mathcal{P}_{A^{0,\bullet}(X)}$ whose homotopy category is equivalent to $D^b_{\text{Coh}}(X)$. Using the theory of superconnections defined in [Qui85], it's natural to extend the classical constructions for holomorphic bundles to cohesive modules. The organization of the paper is as follows.

In section 2, we review the theory of cohesive modules and then construct the analog of Chern connections on a Hermitian cohesive module in the following sense.

Theorem 1.1. *Given a cohesive module $(E^\bullet, \mathbb{E}'')$ with a Hermitian structure h_E, there exist an unique ∂^X-superconnection $\mathbb{E}''$ of total degree -1 such that the superconnection $\mathbb{E} = \mathbb{E}'' + \mathbb{E}'$ is unitary. That is, $\mathbb{E}$ satisfies:*

$$(-1)^{|s|} d^X h_E(s,t) = -h_E(\mathbb{E}s, t) + h_E(s, \mathbb{E}t) \qquad (1.1)$$

for all $s, t \in \mathcal{A}^\bullet(X, E^\bullet)$.

We then study the basic properties of the curvature $\mathcal{R}^{\mathbb{E}}$ of the Chern superconnection $\mathbb{E}$. Though it is no longer a $(1, 1)$ form on X, if we define a new grading on $\mathcal{A}^{p,q}(X, \mathrm{End}^d(E^\bullet))$ by $-p + q + d$, then the curvature $\mathcal{R}$ is of exotic degree zero. Applying the Chern-Weil theory for superconnections, we obtain characteristic forms with values in Bott-Chern cohomology, which is a refinement of deRham cohomology. We prove the deRham cohomology classes of a cohesive module only depends on the $\mathbb{Z}_2$-graded topological bundle structure by transgressing the characteristic forms defined by Chern superconnection to forms defined by the connection component.

In section 3, we prove the characteristic classes in Bott-Chern cohomology are independent of the Hermitian metric by establishing several transgression formulas. These formulas were first obtained by Bott and Chern in [BC65]. To generalize them to cohesive modules, we study the universal cohesive module $\widetilde{E}$ on the space of Hermitian metrics $\mathcal{M}$ on E.

Theorem 1.2 (Bott-Chern transgression formula.)**.** *For any convergent power series $f(T)$, the $\mathcal{M}$-directional derivative of the Bott-Chern characteristic form $\mathrm{Tr}_s f(\mathcal{R}_h)$ is given by:*

$$d^{\mathcal{M}} \mathrm{Tr}_s f(\mathcal{R}_h) = \partial^X \bar{\partial}^X \mathrm{Tr}_s(f'(\mathcal{R}_h) \cdot \theta) \qquad (1.2)$$

where θ is the Maruer Cartan form on $\mathcal{M}$ defined by $\theta = h^{-1} \cdot d^{\mathcal{M}} h$.

The forms $\mathrm{Tr}_s(f'(\mathcal{R}_h) \cdot \theta)$ appeared in the above double transgression formula is the holomorphic analog of Chern-Simons forms. In [Don87], Donaldson studied these secondary characteristic forms their relation to stability. Following Donaldson, the technical computations in section 3 will establish the following formula which generalizes Donaldson's result.

Theorem 1.3 (Donaldson transgression formula for secondary class)**.** *If $g(T)$ is a convergent power series in T, the $\mathcal{M}$-directional derivative of the secondary forms $\mathrm{Tr}_s(g(\mathcal{R}_h) \cdot \theta)$ is given by*

$$d^{\mathcal{M}} \mathrm{Tr}_s(g(\mathcal{R}_h) \cdot \theta) = \frac{1}{2} \bar{\partial}^X \mathrm{Tr}_s(g(\mathcal{R}_h; [\mathbb{E}'_h, \theta]) \cdot \theta) - \frac{1}{2} \partial^X \mathrm{Tr}_s(g(\mathcal{R}_h; [\mathbb{E}'', \theta]) \cdot \theta)$$

$$(1.3)$$

In the last section, we prove the invariance of characteristic classes under homotopy equivalences between cohesive modules. By a criteria proved by

Block in [Blo10], two cohesive modules E, F are homotopy equivalent if and only if there is a degree zero closed morphism $\phi \in \mathcal{P}^0_{\mathcal{A}^{0,\bullet}(X)}(E, F)$ that induces a quasi-isomorphism between $(E^\bullet, \mathbb{E}''_0)$ and $(F^\bullet, \mathbb{F}''_0)$. Using this result, we prove the invariance in two steps. First, we show that characteristic classes of an acyclic cohesive module are trivial.

Proposition 1.4. *Assume $(E^\bullet, \mathbb{E}'')$ is a cohesive module such that $(E^\bullet, \mathbb{E}''_0)$ is an acyclic complex. Let $\mathbb{E}''_t = \sum_k t^{(1-k)/2} \mathbb{E}''_k$ be the rescaled cohesive structure with parameter $t \in \mathbb{R}^+$ with associated curvature $\mathcal{R}_t$ and N_E be the grading operator on E. Then the integral*

$$\mathcal{I}_E = \int_1^\infty \mathrm{Tr}_s\{\exp(-\mathcal{R}_t) \cdot N_E \frac{dt}{t}\} \tag{1.4}$$

is finite and we have

$$\mathrm{ch}(E) = \mathrm{Tr}_s \exp(-\mathcal{R}) = \partial^X \bar{\partial}^X \mathcal{I}_E \tag{1.5}$$

Then we show that the characteristic classes are additive with respect to short exact sequences of mapping cone.

Proposition 1.5. *If $0 \to E \xrightarrow{\phi} F \to \mathrm{Cone}(\phi)$ is the mapping cone sequence for a morphism $\phi \in \mathcal{P}_{\mathcal{A}^{0,\bullet}(X)}(E, F)$, then the Bott-Chern cohomology classes are additive in the sense that the equality*

$$f(E^\bullet, \mathbb{E}'') - f(F^\bullet, \mathbb{F}'') + f(\mathrm{Cone}^\bullet(\phi), \mathbb{C}_\phi) = 0 \tag{1.6}$$

holds in Bott-Chern cohomology.

As a corollary, the characteristic classes defined for cohesive modules descend to $D^b_{\mathrm{Coh}}(X)$ and extend Bott-Chern cohomology to coherent sheaves.

2 Cohesive Modules and Chern superconnections

2.1 Dg-category of cohesive modules

Let X be a complex compact manifold, and let $(\mathcal{A}^{0,\bullet}(X), \bar{\partial}^X)$ be its Dolbeault differential graded algebra (dga). $\mathcal{P}_{\mathcal{A}^{0,\bullet}(X)}$ is the dg-category of cohesive

modules over $(\mathcal{A}^{0,\bullet}(X), \bar{\partial}^X)$. We recall the definition of this dg-category from [Blo10] below. Through this article, we work with double or triple $\mathbb{Z}$-graded objects and we write $|\bullet|$ for the total degree. The commutators and traces are taken in the sense of superspaces.

Definition 2.1. A cohesive module $E = (E^\bullet, \mathbb{E}'')$ on X consists of two data:

1. A finite dimensional $\mathbb{Z}$-graded complex vector bundle $E^\bullet$.

2. A flat $\mathbb{Z}$-graded $\bar{\partial}^X$-superconnection $\mathbb{E}''$ on $E^\bullet$. That is a $\mathbb{C}$-linear map $\mathbb{E}'' : \mathcal{A}^{0,\bullet}(X, E^\bullet) \to \mathcal{A}^{0,\bullet}(X, E^\bullet)$ of total degree 1 which satisfies both the $\bar{\partial}^X$-Leibniz formula:

$$\mathbb{E}''(e \otimes \omega) = (\mathbb{E}''e) \wedge \omega + (-1)^{|e|} e \otimes \bar{\partial}^X \omega, \forall \omega \in \mathcal{A}^{0,\bullet}(X), e \in \mathcal{A}^0(X, E^\bullet) \tag{2.1}$$

and the flatness equation:

$$\mathbb{E}'' \circ \mathbb{E}'' = 0 \tag{2.2}$$

Remark 2.2. In equation (2.1), we view the space of smooth sections of $E^\bullet$ as a right module over $\mathcal{A}^0(X)$. If we consider it as a left module by the isomorphism $I(e \otimes \omega) = (-1)^{|e||\omega|} \omega \otimes e$, the induced superconnection $I(\mathbb{E}'')$ satisfies the usual Leibniz formula:

$$I(\mathbb{E}'')(\omega \otimes e) = \bar{\partial}^X \omega \otimes e + (-1)^{|\omega|} \omega \wedge I(\mathbb{E}'')e \tag{2.3}$$

We continue to work with the right module convention as in [Blo10] so that the shift operation on the dg-category is simply taking $(E^\bullet, \mathbb{E})$ to $(E^{\bullet+1}, -\mathbb{E})$.

Definition 2.3. The objects in the dg-category $\mathcal{P}_{\mathcal{A}^{0,\bullet}(X)}$ are cohesive modules defined above. The degree k morphisms $\mathcal{P}^k_{\mathcal{A}^{0,\bullet}(X)}(E, F)$ between two cohesive modules $E = (E^\bullet, \mathbb{E}'')$, $F = (F^\bullet, \mathbb{F}'')$ are $\mathcal{A}^{0,\bullet}(X)$-linear maps $\phi : \mathcal{A}^{0,\bullet}(X, E^\bullet) \to \mathcal{A}^{0,\bullet}(X, F^\bullet)$ of total degree k. The differential $d : \mathcal{P}^k_{\mathcal{A}^{0,\bullet}(X)}(E, F) \to \mathcal{P}^{k+1}_{\mathcal{A}^{0,\bullet}(X)}(E, F)$ is defined by the commutator:

$$d(\phi) = \mathbb{F}'' \circ \phi - (-1)^k \phi \circ \mathbb{E}'' \tag{2.4}$$

It's simple to verify $d^2 = 0$ and therefore $\mathcal{P}_{\mathcal{A}^{0,\bullet}(X)}$ is a dg-category.

Example 2.4. If $(E^\bullet, \delta)$ is a complex of holomorphic vector bundles on X, it defines a cohesive module with cohesive structure defined by setting $\mathbb{E}_0'' = \delta$, $\mathbb{E}_1'' = (-1)^\bullet \bar{\partial}^{E^\bullet}$ and $\mathbb{E}_k'' = 0$ for $k > 1$. The flatness condition is equivalent to the following set of equations:

$$\delta \circ \delta = 0 \qquad\qquad (\delta \text{ defines a complex of vector bundles}) \qquad (2.5)$$

$$\delta \circ \bar{\partial}^{E^\bullet} = \bar{\partial}^{E^{\bullet+1}} \circ \delta \qquad\qquad (\delta \text{ are holomoprhic homomorphisms}) \qquad (2.6)$$

$$\bar{\partial}^{E^\bullet} \circ \bar{\partial}^{E^\bullet} = 0 \qquad (\bar{\partial}^{E^\bullet} \text{ defines a holomorphic structure on } E^\bullet) \qquad (2.7)$$

Remark 2.5. More generally, Block proved that the homotopy category of $\mathcal{P}_{\mathcal{A}^{0,\bullet}(X)}$ is equivalent to $D_{\mathrm{Coh}}^b(X)$ in [Blo10]. That is, any complex of sheaves of $\mathcal{O}_X$-modules with coherent cohomology is represented by a cohesive module. In particular, any coherent sheaf is represented by $(E^\bullet, \mathbb{E}'')$ such that the underlying complex $(E^\bullet, \mathbb{E}_0'')$ is exact except at the end.

Definition 2.6. A Hermitian form h on a cohesive module E is a Hermitian form on the $\mathbb{Z}$-graded bundle $E^\bullet$ such that the E^k is orthogonal to E^l if $k \neq l$. We denote by $(E, \mathbb{E}'', h)$ a Hermitian cohesive module.

We define a conjugate linear involution on differential forms that is different from complex conjugation. It is motivated by the involution on Clifford algebras.

Definition 2.7. For $\omega \in \mathcal{A}^k(X)$ a complex differential form in degree k, we define

$$\omega^* = (-1)^{\frac{(k+1)k}{2}} \bar{\omega} \qquad (2.8)$$

Lemma 2.8. *The $*$-operation is the unique operation on $\mathcal{A}^\bullet(X)$ such that $f^* = \bar{f}$ if f is a smooth function, $\omega^* = -\bar{\omega}$ if ω is one form and in general,*

$$(\omega \wedge \eta)^* = \eta^* \wedge \omega^*, \forall \omega, \eta \in \mathcal{A}^\bullet(X) \qquad (2.9)$$

Definition 2.9. We extend the $*$-operation to $\mathcal{A}^\bullet(X, \mathrm{End}^\bullet E)$ by

$$(L \otimes \omega)^* = (-1)^{|L||\omega|} L^* \otimes \omega^* \qquad (2.10)$$

for all $L \in \mathcal{A}^0(X, \mathrm{End}^\bullet E)$ and $\omega \in \mathcal{A}^\bullet(X)$.

Lemma 2.10. *The operator $*$ extends to a conjugate linear involution on the algebra $\mathcal{A}^\bullet(X, \mathrm{End}^\bullet E)$.*

Definition 2.11. If h is a Hermitian structure on the cohesive module $E^\bullet$, then we extend h to $\mathcal{A}^\bullet(X, E^\bullet)$ by the formula:

$$h(e \otimes \omega, f \otimes \eta) = \omega^* \wedge h(e, f) \wedge \eta \tag{2.11}$$

where $e, f \in \mathcal{A}^0(X, E^\bullet)$ and $\omega, \eta \in \mathcal{A}^\bullet(X)$.

The following result shows that $*$ defines the adjoint operation with respect to h.

Lemma 2.12. *If $A \in \mathcal{A}^{0,\bullet}(X, \operatorname{End}^\bullet E)$, then $A^* \in \mathcal{A}^{\bullet,0}(X, \operatorname{End}^{-\bullet} E)$ and for any $s, t \in \mathcal{A}^\bullet(X, E^\bullet)$, we have*

$$h(As, t) = h(s, A^*t) \tag{2.12}$$

Proof. Assume $A = L \otimes \tau, s = e \otimes \omega, t = f \otimes \eta$ and $|e| + |L| = |f|$ for otherwise both side of the equation are zero.

$$\begin{aligned}
h(As, t) &= h(L \otimes \tau(e \otimes \omega), f \otimes \eta) = (-1)^{|\tau||e|} h(Le \otimes \tau \wedge \omega, f\eta) \\
&= (-1)^{|\tau||e|}(\tau \wedge \omega)^* h(Le, f)\eta = (-1)^{|\tau||e|}\omega^* \wedge \tau^* h(e, L^*f)\eta \\
&= (-1)^{|\tau||e|}\omega^* h(e, L^*f)\tau^* \wedge \eta = (-1)^{|\tau|(|e|+|f|)}\omega^* h(e, L^* \otimes \tau^*(f \otimes \eta))
\end{aligned}$$

By assumption $|e| + |L| = |f|$, we have $|\tau|\,(|e| + |f|) = |\tau|\,|L| \mod 2$. Finally, since $A^* = (-1)^{|L||\tau|}L^* \otimes \tau^*$, we have

$$h(As, t) = (-1)^{2|L||\tau|}h(s, A^*t) = h(s, A^*t) \tag{2.13}$$

$\square$

2.2 Chern superconnection of a Hermitian cohesive module

The holomorphic Chern connection ∇^E for a holomorphic Hermitian vector bundle E is the unique unitary connection on E with $\bar\partial^X$-component given by the Dolbeault differential $\bar\partial^E$. We generalize this construction in this section to Hermitian cohesive modules.

Lemma 2.13. *For a complex Hermitian vector bundle (E, h) with an $\bar\partial^X$-connection ∇'' which is not necessarily flat, there exist an unique ∂^X-connection ∇' such that $\nabla = \nabla' + \nabla''$ is unitary with respect to h. That is, for any $e, f \in \mathcal{A}^\bullet(X, E)$, ∇ satisfies:*

$$d^X h(e, f) = -h(\nabla e, f) + h(e, \nabla f) \tag{2.14}$$

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Proof. The problem is local, so it suffices to construct ∇ and prove its uniqueness locally. Choose a local frame $s = (s_1, s_2, ...s_n)$ on E, let $\Theta \in \mathcal{A}^{0,1}(X, \mathrm{End} E)$ be the connection $(0,1)$-form such that $\nabla'' s = s \otimes \Theta$. With respect to the frame s, the Hermitian metric h is represented by the Hermitian matrix valued function $\mathcal{H} = h(s, s)$. Since any $(1,0)$-connection ∇' is locally represented by its connection $(1,0)$-form $\Omega \in \mathcal{A}^{1,0}(X, \mathrm{End} E)$ such that $\nabla' s = s \otimes \Omega$, if we set $\nabla = \nabla' + \nabla''$, the condition for ∇ being unitary is:

$$d^X \mathcal{H} = -(\Theta + \Omega)^* \mathcal{H} + \mathcal{H}(\Theta + \Omega) \tag{2.15}$$

Comparing the $(1,0)$ and $(0,1)$ component of equation (2.15) above, then (2.14) is equivalent to

$$\partial^X \mathcal{H} = -\Theta^* \mathcal{H} + \mathcal{H}\Omega \tag{2.16}$$

$$\bar{\partial}^X \mathcal{H} = -\Omega^* \mathcal{H} + \mathcal{H}\Theta \tag{2.17}$$

Using (2.16), we can solve for Ω as

$$\Omega = \mathcal{H}^{-1} \partial^X \mathcal{H} + \mathcal{H}^{-1} \Theta^* \mathcal{H} \tag{2.18}$$

which shows the uniqueness of ∇'. But we can also use (2.18) as the definition of Ω. It remains to verify that with Ω so defined, equation (2.17) is satisfied. Since $\mathcal{H}$ is a Hermitian matrix, $\mathcal{H}^* = \mathcal{H}$, we can now compute Ω^* as

$$\begin{aligned} \Omega^* =& (\partial^X \mathcal{H})^* (\mathcal{H}^{-1})^* + \mathcal{H}^* \Theta^{**} (\mathcal{H}^{-1})^* \\ =& -\bar{\partial}^X \mathcal{H}^* (\mathcal{H}^*)^{-1} + \mathcal{H}^* \Theta (\mathcal{H}^*)^{-1} \\ =& -\bar{\partial}^X \mathcal{H} \mathcal{H}^{-1} + \mathcal{H}\Theta \mathcal{H}^{-1} \end{aligned} \tag{2.19}$$

Multiplying $\mathcal{H}$ on the right, we get (2.17). $\qquad\square$

Remark 2.14. The above equation (2.14) differs by a minus sign from the ordinary equation for unitary connections:

$$d^X h(e, f) = h(\nabla e, f) + h(e, \nabla f) \tag{2.20}$$

This is caused by our extension of h to $\mathcal{A}^\bullet(X, E^\bullet)$. In the ordinary case, if $s \otimes \Omega$ is a one form with values in E, then $h(s \otimes \Omega, t) = \overline{\Omega} h(s, t)$. However, by our definition, $h(s \otimes \Omega, t) = \Omega^* h(s, t) = -\overline{\Omega} h(s, t)$.

Definition 2.15. We define an exotic grading on the spaces $\mathcal{A}^\bullet(X, \mathrm{End}^\bullet E)$ and $\mathcal{A}^\bullet(X, E^\bullet)$. For an element $A = L \otimes \tau$, where $L \in \mathrm{End}^d(E^\bullet)$ or $L \in E^d$, and $\tau \in \mathcal{A}^{p,q}(X)$, define its exotic degree by $\deg(A) = -p + q + d$. $\mathcal{A}^\bullet(X)$ inherit an exotic grading as well given by $\deg(\tau) = -p + q$. We denote by $\mathcal{G}^k$ the subspaces of exotic degree k. In particular, the subspace $\mathcal{G}^0$ in $\mathcal{A}^\bullet(X)$ consists of forms with bi-degree (p, p).

Lemma 2.16. *With respect to the exotic grading* $\deg$, $\mathcal{A}^\bullet(X)$ *is a* $\mathbb{Z}$*-graded algebra;* $\mathcal{A}^\bullet(X, \mathrm{End}^\bullet E)$ *is a* $\mathbb{Z}$*-graded algebra and* $\mathbb{Z}$*-graded module over* $\mathcal{A}^\bullet(X)$; $\mathcal{A}^\bullet(X, E^\bullet)$ *is both a* $\mathbb{Z}$*-graded module over* $\mathcal{A}^\bullet(X)$ *where the action is exterior product, and a* $\mathbb{Z}$*-graded module over* $\mathcal{A}^\bullet(X, \mathrm{End}^\bullet E)$ *where the action is evaluation. In addition, the* $*$*-operator defined before on* $\mathcal{A}^\bullet(X)$ *and* $\mathcal{A}^\bullet(X, \mathrm{End}^\bullet E)$ *maps* $\mathcal{G}^\bullet$ *to* $\mathcal{G}^{-\bullet}$.

Proof. It's simple to verify that the subspaces $\mathcal{G}^\bullet$ respects all these structures in the sense

$$\mathcal{G}^i \cdot \mathcal{G}^j \subseteq \mathcal{G}^{i+j} \tag{2.21}$$

whenever the composition is defined by the algebra multiplication or module action. $\qquad\square$

Proposition 2.17 (Chern superconnection). *Let* $E = (E^\bullet, \mathbb{E}'', h)$ *be a Hermitian cohesive module. There exist an unique* ∂^X*-superconnection* $\mathbb{E}'$: $\mathcal{A}^0(X, E^\bullet) \to \mathcal{A}^{\bullet,0}(X, E^\bullet)$ *of exotic degree* -1 *such that the superconnection* $\mathbb{E} = \mathbb{E}' + \mathbb{E}''$ *is unitary. That is for any* $s, t \in \mathcal{A}^\bullet(X, E^\bullet)$, *we have*

$$(-1)^{|s|} d^X h(s, t) = -h(\mathbb{E}s, t) + h(s, \mathbb{E}t) \tag{2.22}$$

Proof. There is a decomposition of $\mathbb{E}'' = \mathbb{E}''_1 + \sum_{k \neq 1} \mathbb{E}''_k$ into an ordinary $\bar{\partial}^X$-connection $\mathbb{E}''_1$ on $E^\bullet$ and linear terms $\mathbb{E}''_k \in \mathcal{A}^{0,k}(X, \mathrm{End}^{1-k} E)$.

By lemma 2.13, there is an unique ∂^X-connection $\mathbb{E}'_1$ on $E^\bullet$ such that $\nabla^{E^\bullet} = \mathbb{E}''_1 + \mathbb{E}'_1$ is unitary on the graded Hermitian complex vector bundle $E^\bullet$. If we set $\mathbb{E}'_k = (\mathbb{E}''_k)^*$ for $k \neq 1$, by lemma 2.12, $\mathbb{E}_k = \mathbb{E}''_k + \mathbb{E}'_k$ is unitary. Adding $\nabla^{E^\bullet}$ and $\mathbb{E}_k$, the resulting d^X-superconnection $\mathbb{E}$ is unitary. Uniqueness is obvious from the construction and equation (2.22). $\qquad\square$

Remark 2.18. We can always write a homogeneous term $A = L \otimes \tau$ in such a way that τ is a real form of degree k. Since A is odd, A^* is just $L^* \otimes \tau^*$

and therefore $A = A^*$ is equivalent to $L = (-1)^{\frac{k(k+1)}{2}} L^*$. In other words, L satisfies the following conditions:

$$\begin{cases} L \text{ is Hermitian} & k = 0, 3 \ (\text{mod } 4) \\ L \text{ is skew Hermitian} & k = 1, 2 \ (\text{mod } 4) \end{cases}$$

In particular, this is compatible with the well known fact that an unitary connection has skew Hermitian connection matrix.

2.3   Chern-Weil construction of characteristic forms

Definition 2.19. Let $\mathbb{E}$ be the superconnection for a Hermitian cohesive module $(E^\bullet, \mathbb{E}'', h)$ given by Proposition 2.17. We call it the Chern superconnection. The curvature of $\mathbb{E}$ is defined by the usual formula

$$\mathcal{R} = \mathbb{E}^2 = \frac{1}{2}[\mathbb{E}, \mathbb{E}] \tag{2.23}$$

In the classical case of Hermitian holomorphic vector bundles, the curvature of the Chern connection is a $(1, 1)$-form. In the case of Hermitian cohesive modules, we have similar properties of Chern superconnections for cohesive modules.

Lemma 2.20. *If $\mathbb{E}$ is the Chern superconnection for a Hermitian cohesive module $(E^\bullet, \mathbb{E}, h)$ and $\mathcal{R}$ is its curvature, then it satisfies the following properties:*

$$\mathcal{R}^* = \mathcal{R} \tag{2.24}$$
$$\mathbb{E}' \circ \mathbb{E}' = 0 \tag{2.25}$$
$$\mathcal{R} = [\mathbb{E}', \mathbb{E}''] \tag{2.26}$$

Consequently, the curvature $\mathcal{R}$ is of exotic degree zero $\mathcal{R} \in \mathcal{G}^0$.

Proof. By definition, the curvature is

$$\begin{aligned} \mathcal{R} =& \mathbb{E} \circ \mathbb{E} = (\mathbb{E}' + \mathbb{E}'') \circ (\mathbb{E}' + \mathbb{E}'') \\ =& (\mathbb{E}')^2 + [\mathbb{E}', \mathbb{E}''] + (\mathbb{E}'')^2 \\ =& (\mathbb{E}')^2 + [\mathbb{E}', \mathbb{E}''] \end{aligned}$$

where the term $(\mathbb{E}'')^2$ vanishes by our flatness assumption on $\mathbb{E}''$. Since the exotic degree of $\mathbb{E}'$ is -1 and that of $[\mathbb{E}', \mathbb{E}'']$ is 0, $(\mathbb{E}')^2 \in \mathcal{G}^{-2}$ and $[\mathbb{E}', \mathbb{E}''] \in \mathcal{G}^0$. We use equation (2.22) repeatedly to get:

$$
\begin{aligned}
0 = & (d^X)^2 h(s,t) = (-1)^{|s|+1} d^X h(\mathbb{E}s, t) + (-1)^{|s|} d^X h(s, \mathbb{E}t) \\
= & - h(\mathcal{R}s, t) + h(\mathbb{E}s, \mathbb{E}t) - h(\mathbb{E}s, \mathbb{E}t) + h(s, \mathcal{R}t) \\
= & - h(\mathcal{R}s, t) + h(s, \mathcal{R}t)
\end{aligned}
$$

Since $\mathcal{R}$ is linear, by Lemma 2.12, the curvature $\mathcal{R}$ is self adjoint $\mathcal{R} = \mathcal{R}^*$. Since $*$ maps $\mathcal{G}^{-2}$ to $\mathcal{G}^2$, we see $(\mathbb{E}')^2 = 0$. The last equality follows from this. $\qquad \square$

Lemma 2.21 (Bianchi Identity). $[\mathbb{E}, \mathcal{R}] = 0$

Proof. By definition, $\mathcal{R} = \frac{1}{2}[\mathbb{E}, \mathbb{E}]$. Using either the graded Jacobi identity or simply by expanding the expression, we have

$$
[\mathbb{E}, \mathcal{R}] = [\mathbb{E}, \mathbb{E}^2] = \mathbb{E} \cdot \mathbb{E}^2 - \mathbb{E}^2 \cdot \mathbb{E} = 0
$$

$\qquad \square$

Definition 2.22. Assume (E, h) is a Hermitian cohesive module with Chern superconnection $\mathbb{E}_h$ and curvature $\mathcal{R}_h$. Let $f(T)$ be a convergent power series in T, we define the characteristic form of (E, h) associated to $f(T)$ by

$$
f(E^\bullet, \mathbb{E}'', h) = \operatorname{Tr}_s f(\mathcal{R}_h) \tag{2.27}
$$

where Tr_s is the supertrace in the sense of Quillen.

Remark 2.23. The power series is required to be convergent since $\mathcal{R}$ is no longer concentrated in degree $(1, 1)$.

Proposition 2.24. *The characteristic forms are closed*

$$
d^X f(E^\bullet, \mathbb{E}'', h) = 0 \tag{2.28}
$$

Proof. Without loss of generality, we assume $f(T) = T^n$ is a monomial. Then we have

$$
d^X \operatorname{Tr}_s(\mathcal{R}_h^n) = \sum_i \operatorname{Tr}_s(\mathcal{R}_h^{i-1}[\mathbb{E}_h, \mathcal{R}_h]\mathcal{R}_h^{n-i}) \tag{2.29}
$$

Since each term in the summation is zero by Bianchi identity, $\operatorname{Tr}_s(\mathcal{R}_h^n)$ is closed. $\qquad \square$

Consequently, the characteristic forms defines deRham cohomology classes. We will show that these cohomology classes only dependent on the connection component $\nabla^E = \mathbb{E}_1$ in $\mathbb{E}_h$. Let $\mathbb{E}_t = (1-t)\nabla^E + t\mathbb{E}$ be a one parameter family of superconnections that joins ∇^E to $\mathbb{E}$ where $\nabla^E = \mathbb{E}_1$ is the connection term of $\mathbb{E}$. If we write $A = \mathbb{E} - \mathbb{E}_1 = \sum_{k \neq 1} \mathbb{E}_k$ for the linear terms, then $\mathbb{E}_t = \nabla^E + tA$. Let $\mathcal{R}_t$ be the corresponding curvatures for $\mathbb{E}_t$.

Lemma 2.25. *The deformation of curvature is computed by*

$$\frac{d}{dt}\mathcal{R}_t = [\mathbb{E}_t, A] \tag{2.30}$$

Proof. Since $\mathbb{E}_t = \nabla^E + tA$, we have $\mathcal{R}_t = \mathcal{R}_0 + t[\nabla^E, A] + t^2 A^2$. We can then compute

$$\frac{d}{dt}\mathcal{R}_t = [\nabla^E, A] + 2tA^2 \tag{2.31}$$

since $2tA^2 = t[A, A] = [tA, A]$, the right side of the above equality is $[\mathbb{E}_t, A]$. $\square$

Corollary 2.26. *If we define $f(E, \mathbb{E}_t)$ by the same formula $f(\mathcal{R}_t)$, we have*

$$f(E, \mathbb{E}) - f(E, \nabla^E) = d^X \int_0^1 \mathrm{Tr}_s\{A \cdot f'(\mathcal{R}_t)\}dt \tag{2.32}$$

Consequently, $f(E, \mathbb{E})$ is cohomologous to $f(E, \nabla^E)$ in deRham cohomology.

It is well-known that $f(E, \nabla^E)$ only depends on the topological vector bundle structure of E as a $\mathbb{Z}_2$-graded vector bundle, so the characteristic class in deRham cohomology defined by $f(E^\bullet, h, \mathbb{E}'')$ is independent of h.

Definition 2.27. Let $\mathcal{A}^{p,p}(X)$ be the space of forms of bi-degree (p, p) and $Z^p(X)$ be the subspace of d^X-closed forms. We set $B^p(X)$ the subspace of $Z^p(X)$ that is the image of $\mathcal{A}^{p-1,p-1}(X)$ under $\partial^X \bar{\partial}^X$, then the p-th Bott-Chern cohomology $H^p_{BC}(X)$ is defined by:

$$H^p_{BC}(X) = \frac{Z^p(X)}{B^p(X)}$$

By Lemma 2.20, $f(E, h) \in \mathcal{G}^0$ and hence the forms defines Bott-Chern cohomology classes. We will prove in the next section that these refined cohomology classes are independent of the Hermitian structure h. In the last section, we will show they are further invariant under homotopy equivalences between cohesive modules.

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3 Transgression formulas for characteristic forms

3.1 Universal cohesive module and Maurer-Cartan equation

We fix a cohesive module $E = (E^\bullet, \mathbb{E}'')$ on X and let $\mathcal{M}$ be the space of Hermitian structures on $E^\bullet$. We endow it with the topology of uniform $\mathcal{C}^\infty$-convergence on compact subsets of X. If $h \in \mathcal{M}, x \in X$, let A_x^h be the subset of $\operatorname{End}^0 E_x$ whose elements are Hermitian endomorphisms with respect to h_x. A_x^h forms a bundle A^h over X, and the tangent space of $\mathcal{M}$ at h, $T_h\mathcal{M}$, can be identified with the linear space of sections of A^h over X.

Definition 3.1. Consider the projection $\pi : \mathcal{M} \times X \to X$, we define $\widetilde{E}^\bullet$ as the pull-back bundle $\pi^* E^\bullet$ over $\mathcal{M} \times X$. There is an universal cohesive structure $\widetilde{\mathbb{E}}''$ on $\widetilde{E}^\bullet$ defined by

$$\widetilde{\mathbb{E}}'' = \bar{\partial}^{\mathcal{M}} + \mathbb{E}'' \tag{3.1}$$

Finally, we write $\tilde{h}$ for the universal Hermitian form on $\widetilde{E}^\bullet$. We denote by $\widetilde{E}$ the universal Hermitian cohesive module $(\widetilde{E}^\bullet, \widetilde{\mathbb{E}}'', \tilde{h})$.

Remark 3.2. The bundle $\widetilde{E}^\bullet$ is flat in the $\mathcal{M}$-direction with flat connection $d^{\mathcal{M}}$. In the above formula, $\bar{\partial}^{\mathcal{M}}$ acts on an element $e \otimes \omega$ by $(-1)^{|e|} e \otimes \bar{\partial}^{\mathcal{M}} \omega$.

Lemma 3.3. $\widetilde{\mathbb{E}}''$ *defines a $\mathbb{Z}$-graded flat $\bar{\partial}_{\mathcal{M} \times X}$-connection on $\widetilde{E}^\bullet$.*

Proof. Note that the vertical cohesive structure $\mathbb{E}''$ is constant along $\mathcal{M}$-direction, so $d^{\mathcal{M}}\mathbb{E}'' = 0$, we have

$$(\widetilde{\mathbb{E}}'')^2 = (\bar{\partial}^{\mathcal{M}} + \mathbb{E}'')^2 = (\bar{\partial}^{\mathcal{M}})^2 + \bar{\partial}^{\mathcal{M}}(\mathbb{E}'') + (\mathbb{E}'')^2 = 0 \tag{3.2}$$

$\square$

Definition 3.4. The Maurer-Cartan one form $\theta \in \mathcal{A}^1(\mathcal{M} \times X, \operatorname{End}^0 \widetilde{E})$ on $\mathcal{M}$ with values in A is the one form defined by

$$\theta = h^{-1}d^{\mathcal{M}}h \tag{3.3}$$

where d^M is the exterior differential on $\mathcal{M}$.

Lemma 3.5. θ *is a one form in the $\mathcal{M}$-direction with values in Hermitian endomorphisms, $\theta \in \mathcal{A}^1(\mathcal{M}, \operatorname{End}(E, h))$.*

Proof. We choose a frame $(e_1, e_2, ..., e_n)$ for $E^\bullet$ locally on X. Any Hermitian form h is represented by a smooth function with values in Hermitian matrix by

$$\mathcal{H} : x \in X \to [h_x(e_i, e_j)]_{ij} \tag{3.4}$$

If $\mathcal{H}_t(x)$ is a smooth family of Hermitian matrix valued functions with parameter t, we can differentiate $\mathcal{H}_t(X)$ with respect to t, then $\mathcal{H}_t^{-1}\dot{\mathcal{H}}_t$ is a Hermitian with respect to $\mathcal{H}_t$. $\square$

Lemma 3.6. θ *satisfies the Maurer-Cartan equation:*

$$d^{\mathcal{M}}\theta = -\theta^2 \tag{3.5}$$

Proof. This is straightforward calculation:

$$d^{\mathcal{M}}\theta = d^{\mathcal{M}}(h^{-1}d^{\mathcal{M}}h) = d^{\mathcal{M}}(h^{-1})d^{\mathcal{M}}h + h^{-1}d^{\mathcal{M}}(d^{\mathcal{M}}h) \tag{3.6}$$
$$= -h^{-1}(d^{\mathcal{M}}h)h^{-1}d^{\mathcal{M}}h = -\theta^2 \tag{3.7}$$

$\square$

Since $\widetilde{E}$ is flat along the $\mathcal{M}$-direction, by the explicit expression of θ with respect to a local frame on X, we see θ measures the deformation of h over $\mathcal{M}$ in the following sense.

Lemma 3.7. *The flat $\mathcal{M}$-directional exterior derivative $d^{\mathcal{M}}$ has the following compatibility relation with the universal Hermitian form $\tilde{h}$*

$$(-1)^{|s|}d^{\mathcal{M}}\tilde{h}(s,t) = -\tilde{h}(d^{\mathcal{M}}s, t) + \tilde{h}(s, \theta t) + \tilde{h}(s, d^{\mathcal{M}}t) \tag{3.8}$$

for all $s, t \in \mathcal{A}^0(\mathcal{M} \times X, \widetilde{E})$.

The above lemma together with equation (2.22) for each $h \in \mathcal{M}$ shows that $\widetilde{\mathbb{E}} = d^{\mathcal{M}} + \mathbb{E}_h$ differs from the Chern superconnection of $(\widetilde{E}^\bullet, \tilde{h}, \widetilde{\mathbb{E}}'')$ by the Maurer-Cartan form θ.

Proposition 3.8. *The universal superconnection $\widetilde{\mathbb{E}}$ satisfies the following equation*

$$(-1)^{|s|}d^{\mathcal{M}\times X}\tilde{h}(s,t) = -\tilde{h}(\widetilde{\mathbb{E}}s, t) + \tilde{h}(s, \widetilde{\mathbb{E}}t) + \tilde{h}(s, \theta t) \tag{3.9}$$

Even though the previous proposition shows $\widetilde{\mathbb{E}}$ is not the Chern super-connection, we will use $\widetilde{\mathbb{E}}$ to study $\widetilde{E}^\bullet$. Denote by $\widetilde{\mathcal{R}}$ the curvature of $\widetilde{\mathbb{E}}$, it is given by

$$\widetilde{\mathcal{R}} = (d^{\mathcal{M}} + \mathbb{E}_h) \circ (d^{\mathcal{M}} + \mathbb{E}_h) = d^{\mathcal{M}}\mathbb{E}_h + \mathcal{R}_h \tag{3.10}$$

Proposition 3.9. *The $\mathcal{M}$-directional derivative of the Chern superconnection $\mathbb{E}_h$ is given by*

$$d^{\mathcal{M}}\mathbb{E}_h = -[\mathbb{E}'_h, \theta] \tag{3.11}$$

and the $\mathcal{M}$-directional derivative of curvature $\widetilde{\mathcal{R}}$ is given by

$$d^{\mathcal{M}}\widetilde{\mathcal{R}} = d^{\mathcal{M}}\mathcal{R}_h \tag{3.12}$$

Proof. Since $\mathbb{E}_h = \mathbb{E}'_h + \mathbb{E}''$ and $\mathbb{E}''$ is independent of h, we have

$$d^{\mathcal{M}}\mathbb{E}_h = d^{\mathcal{M}}\mathbb{E}'_h \tag{3.13}$$

By the explicit construction of $\mathbb{E}'_h$ from $\mathbb{E}''$ and h, we can write

$$\mathbb{E}'_h = h^{-1} \circ \mathbb{E}'' \circ h \tag{3.14}$$

Taking exterior differential in $\mathcal{M}$-variable, we have

$$
\begin{aligned}
d^{\mathcal{M}}\mathbb{E}'_h &= d^{\mathcal{M}}h^{-1} \circ \mathbb{E}' \circ h - h^{-1} \circ \mathbb{E}' \circ d^{\mathcal{M}}h \\
&= -h^{-1} \circ d^{\mathcal{M}}h \circ h^{-1} \circ \mathbb{E}' \circ h - h^{-1} \circ \mathbb{E}' \circ h \circ h^{-1} \circ d^{\mathcal{M}}h \\
&= -\theta \circ \mathbb{E}'_h - \mathbb{E}'_h \circ \theta = -[\mathbb{E}'_h, \theta]
\end{aligned}
$$

$\square$

3.2 Bott-Chern double transgression formula

Recall the characteristic forms defined in the previous section $f(E^\bullet, \mathbb{E}'', h) = \mathrm{Tr}_s f(\mathcal{R}_h)$ for a convergent power series $f(T)$, the following transgression formula computes the deformation of the characteristic forms over $\mathcal{M}$.

Proposition 3.10 (First Transgression Formula)**.**

$$d^{\mathcal{M}}\mathrm{Tr}_s f(\mathcal{R}_h) = -\bar{\partial}^X \mathrm{Tr}_s(f'(\mathcal{R}_h) \cdot [\mathbb{E}'_h, \theta]) \tag{3.15}$$

$$\partial^X \mathrm{Tr}_s(f'(\mathcal{R}_h) \cdot [\mathbb{E}'_h, \theta]) = 0 \tag{3.16}$$

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Proof. On the universal Hermitian cohesive module $\widetilde{E}^\bullet$, by proposition 2.24 and equation (3.10), we have

$$0 = d^{\mathcal{M} \times X} \mathrm{Tr}_s f(\widetilde{\mathcal{R}}) = d^{\mathcal{M} \times X} \mathrm{Tr}_s f(\mathcal{R}_h + d^{\mathcal{M}} \mathbb{E}_h) \qquad (3.17)$$

Without loss of generality, we assume $f(T) = T^n$ and we expand (3.17) by the multilinear property of $\mathrm{Tr}_s f(T)$ to get:

$$0 = d^X \mathrm{Tr}_s \mathcal{R}_h^n + d^{\mathcal{M}} \mathrm{Tr}_s \mathcal{R}_h^n + d^X \sum_i \mathrm{Tr}_s(\mathcal{R}_h^i \cdot d^{\mathcal{M}} \mathbb{E}_h \cdot \mathcal{R}_h^{n-1-i})\} + \dots \quad (3.18)$$

The first term is zero by proposition 2.24 and the omitted terms are at least degree 2 forms in $\mathcal{M}$-variables. If we collects the forms of degree one in the $\mathcal{M}$-variables, we have

$$0 = d^{\mathcal{M}} \mathrm{Tr}_s(\mathcal{R}_h^n) + n \cdot d^X \mathrm{Tr}_s(\mathcal{R}_h^{n-1} \cdot d^{\mathcal{M}} \mathbb{E}_h) \qquad (3.19)$$

where we commute $\mathcal{R}_h^i \cdot d^{\mathcal{M}} \mathbb{E}_h$ and $\mathcal{R}_h^{n-1-i}$ under Tr_s. Using lemma 3.9, $d^{\mathcal{M}} \mathbb{E}_h = -[\mathbb{E}'_h, \theta]$, we have the following equality:

$$d^{\mathcal{M}} \mathrm{Tr}_s f(\mathcal{R}_h) = d^X \mathrm{Tr}_s(f'(\mathcal{R}_h) \cdot [\mathbb{E}'_h, \theta]) \qquad (3.20)$$

Now we compare both sides of (3.20) and the consider the subspaces $\mathcal{G}^\bullet$ defined by the exotic degree. On the left side of (3.20), we have a one form on $\mathcal{M}$ with values in $\mathcal{G}^0$ since $\mathcal{R}_h \in \mathcal{G}^0$. On the right side, $\mathcal{R}_h$ is in $\mathcal{G}^0$ while $[\mathbb{E}'_h, \theta]$ is in $\mathcal{G}^{-1}$ since $\mathbb{E}'_h$ is of exotic degree -1. Finally, since ∂^X increases the exotic degree by -1 while $\bar{\partial}^X$ increases the exotic degree by 1, we get equation (3.15) by comparing the $\mathcal{G}^0$ component and equation (3.16) by the $\mathcal{G}^{-2}$ component. $\qquad \square$

Our next goal is to express $\mathrm{Tr}_s(f'(\mathcal{R}_h) \cdot [\mathbb{E}'_h, \theta])$ in equation (3.16) as the image of ∂^X. To do this, we first introduce a notation.

Definition 3.11. If $g(T) = T^n$, for a pair $(A; B)$ of variables, we define

$$g(A; B) = \sum_{i=1}^n A^{i-1} B A^{n-i} \qquad (3.21)$$

In general for a convergent power series $g(T)$, we define $g(A; B)$ by the previous formula for its homogeneous components and take the sum. A simple norm estimates shows the convergence.

Proposition 3.12 (Second Transgression formula).

$$\text{Tr}_s(f'(\mathcal{R}_h) \cdot [\mathbb{E}'_h, \theta]) = \partial^X \text{Tr}_s(f'(\mathcal{R}_h) \cdot \theta) \tag{3.22}$$

Proof. For each $\mathbb{E}_h$ along the vertical fiber X, we have

$$d^X \text{Tr}_s(f'(\mathcal{R}_h) \cdot \theta) = \text{Tr}_s([\mathbb{E}_h, f'(\mathcal{R}_h) \cdot \theta]) \tag{3.23}$$

By the graded Leibniz formula of $\mathbb{E}_h$, we can expand the equation and compute

$$\begin{aligned} d^X \text{Tr}_s(f'(\mathcal{R}_h) \cdot \theta) &= \text{Tr}_s([\mathbb{E}_h, f'(\mathcal{R}_h)] \cdot \theta) + \text{Tr}_s(f'(\mathcal{R}_h) \cdot [\mathbb{E}_h, \theta]) \\ &= \text{Tr}_s(f'(\mathcal{R}_h; [\mathbb{E}_h, \mathcal{R}_h]) \cdot \theta) + \text{Tr}_s(f'(\mathcal{R}_h) \cdot [\mathbb{E}_h, \theta]) \end{aligned}$$

By Bianchi identity, the first term is zero, hence we have

$$d^X \text{Tr}_s(f'(\mathcal{R}_h) \cdot \theta) = \text{Tr}_s(f'(\mathcal{R}_h) \cdot [\mathbb{E}'_h, \theta]) + \text{Tr}_s(f'(\mathcal{R}_h) \cdot [\mathbb{E}''_h, \theta]) \tag{3.24}$$

Note as before, $[\mathbb{E}'_h, \theta] \in \mathcal{G}^{-1}$, $[\mathbb{E}''_h, \theta] \in \mathcal{G}^1$ and $\text{Tr}_s(f'(\mathcal{R}_h) \cdot \theta) \in \mathcal{G}^0$, comparing the $\mathcal{G}^{-1}$ component we have (3.23). $\square$

Combining the first transgression formula (3.10) and second transgression formula (3.22), we established the double transgression formula for characteristic forms on cohesive modules.

Corollary 3.13 (Bott-Chern formula). *The $\mathcal{M}$-directional derivative of the characteristic form $f(E, \mathbb{E}'', h)$ at h is given by:*

$$d^{\mathcal{M}} \text{Tr}_s f(\mathcal{R}_h) = \partial^X \bar{\partial}^X \text{Tr}_s(f'(\mathcal{R}_h) \cdot \theta) \tag{3.25}$$

This generalizes the classical formula obtained by Bott and Chern in [BC65] for holomorphic vector bundles. If we view the previous equality in Bott-Chern cohomology, the right hand side is zero and we established the invariance of characteristic classes under metric deformation.

Corollary 3.14. *The characteristic forms $f(E, \mathbb{E}'', h)$ in the Bott-Chern cohomology are independent of the choice of Hermitian metric h.*

In the last section, we will show the characteristic classes only dependent on the homotopy class of the cohesive module.

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3.3 Transgression formula for secondary classes

Our next goal is to study the differential forms $\mathrm{Tr}_s(f'(\mathcal{R}_h) \cdot \theta)$ that appear in the double transgression formula. As we will prove later, they are the secondary characteristic classes and define holomorphic analog of the Chern-Simons forms. We will use them to study infinite determinant bundle and stability in a separate paper. For their applications in holomorphic vector bundles, see [BGS88a][BGS88b][BGS88c].

Lemma 3.15. *The curvature tensors $\mathcal{R}_h$ has $\mathcal{M}$-directional derivative given by:*

$$d^{\mathcal{M}}(\mathcal{R}_h) = [\mathbb{E}_h, [\mathbb{E}'_h, \theta]] \tag{3.26}$$

Proof. By Bianchi identity for the universal cohesive module and use Lemma 3.9, we have:

$$0 = [\widetilde{\mathbb{E}}, \widetilde{\mathcal{R}}] = [d^{\mathcal{M}} + \mathbb{E}_h, \mathcal{R}_h - [\mathbb{E}'_h, \theta]] \tag{3.27}$$

Expand the terms and use lemma (3.9) again, we have

$$0 = d^{\mathcal{M}}\mathcal{R}_h + [\mathbb{E}_h, \mathcal{R}_h] - [\mathbb{E}_h, [\mathbb{E}'_h, \theta]] + (d^{\mathcal{M}})^2(\mathbb{E}_h) \tag{3.28}$$

By Bianchi identity for $\mathbb{E}_h$, the middle term in (3.28) vanishes and we get (3.26). $\square$

We will start to prove the main result of this section, namely the form $\mathrm{Tr}_s(f'(\mathcal{R}_h) \cdot \theta)$ appearing in the double transgression formula is itself a well-defined secondary characteristic form. Like the double transgression formula of Bott and Chern, the goal is to compute the $\mathcal{M}$-directional derivative of $\mathrm{Tr}_s(f'(\mathcal{R}_h) \cdot \theta)$ and show it's in the image of ∂^X and $\bar{\partial}^X$. We break the lengthy computation into several lemmas.

Lemma 3.16.

$$d^X \mathrm{Tr}_s(g(\mathcal{R}_h; [\mathbb{E}'_h, \theta]) \cdot \theta) = \mathrm{Tr}_s(g(\mathcal{R}_h; [\mathbb{E}_h, [\mathbb{E}'_h, \theta]]) \cdot \theta) \tag{3.29}$$
$$+ \mathrm{Tr}_s(g(\mathcal{R}_h; [\mathbb{E}'_h, \theta]) \cdot [\mathbb{E}_h, \theta])$$

Proof. Without loss of generality, we assume $g(T) = T^n$. By our definition, $g(\mathcal{R}_h; [\mathbb{E}_h, [\mathbb{E}'_h, \theta]])$ is a summation of the form:

$$g(\mathcal{R}_h; [\mathbb{E}_h, [\mathbb{E}'_h, \theta]]) = \sum_{i+j=n-1} \mathcal{R}_h^i [\mathbb{E}_h, [\mathbb{E}'_h, \theta]] \mathcal{R}_h^j \tag{3.30}$$

If we consider the following differential, keeping in mind when passing d^X over Tr_s, we act via $\mathbb{E}_h$ and follows the Leibniz rule

$$d^X \sum_i \mathrm{Tr}_s(\mathcal{R}_h^i[\mathbb{E}_h', \theta]\mathcal{R}_h^{n-i-1}\theta) = \sum_i d^X \mathrm{Tr}_s(\mathcal{R}_h^i[\mathbb{E}_h', \theta]\mathcal{R}_h^{n-i-1}\theta) \tag{3.31}$$

$$= \sum_{i+j+k=n-2} \mathrm{Tr}_s(\mathcal{R}_h^i[\mathbb{E}_h, \mathcal{R}_h]\mathcal{R}_h^j[\mathbb{E}_h', \theta]\mathcal{R}_h^k\theta) + \sum_{i=1}^n \mathrm{Tr}_s(\mathcal{R}_h^{i-1}[\mathbb{E}_h, [\mathbb{E}_h', \theta]]\mathcal{R}_h^{n-i}\theta)$$

$$+ \sum_{i+j+k=n-2} \mathrm{Tr}_s(\mathcal{R}_h^i[\mathbb{E}_h', \theta]\mathcal{R}_h^j[\mathbb{E}_h, \mathcal{R}_h]\mathcal{R}_h^k\theta) + \sum_{i=1}^n \mathrm{Tr}_s(\mathcal{R}_h^{i-1}[\mathbb{E}_h', \theta]\mathcal{R}_h^{n-i}[\mathbb{E}_h, \theta])$$

The terms in the first and third summations are zero by Bianchi identity $[\mathbb{E}_h, \mathcal{R}_h] = 0$. Since the second and last summations are just $\mathrm{Tr}_s(g(\mathcal{R}_h; [\mathbb{E}_h, [\mathbb{E}_h', \theta]]) \cdot \theta)$ and $\mathrm{Tr}_s(g(\mathcal{R}_h; [\mathbb{E}_h', \theta]) \cdot [\mathbb{E}_h, \theta])$ respectively, the result follows. $\qquad\square$

Lemma 3.17.

$$d^\mathcal{M}\mathrm{Tr}_s(g(\mathcal{R}_h) \cdot \theta) + \mathrm{Tr}_s(g(\mathcal{R}_h) \cdot \theta^2) = \bar{\partial}^X \mathrm{Tr}_s(g(\mathcal{R}_h; [\mathbb{E}_h', \theta]) \cdot \theta) \tag{3.32}$$
$$- \mathrm{Tr}_s(g(\mathcal{R}_h; [\mathbb{E}_h, \theta]) \cdot [\mathbb{E}'', \theta])$$

$$\mathrm{Tr}_s(g(\mathcal{R}_h; [\mathbb{E}_h', \theta]) \cdot [\mathbb{E}_h', \theta]) = 0 \tag{3.33}$$

$$\mathrm{Tr}_s(g(\mathcal{R}_h; [\mathbb{E}'', \theta]) \cdot [\mathbb{E}'', \theta]) = 0 \tag{3.34}$$

Proof. Without loss of generality, we assume $g(T) = T^n$ is a monomial. By Leibniz formula, we get

$$d^\mathcal{M}\mathrm{Tr}_s(g(\mathcal{R}_h) \cdot \theta) = \mathrm{Tr}_s(g(\mathcal{R}_h; d^\mathcal{M}\mathcal{R}_h) \cdot \theta) + \mathrm{Tr}_s(g(\mathcal{R}_h) \cdot d^\mathcal{M}\theta) \tag{3.35}$$

We can substitute $d^\mathcal{M}\mathcal{R}_h$ and $d^\mathcal{M}\theta$ in the above equation by previous two lemmas 3.6 and 3.15. Then we have

$$d^\mathcal{M}\mathrm{Tr}_s(g(\mathcal{R}_h) \cdot \theta) = \mathrm{Tr}_s(g(\mathcal{R}_h; [\mathbb{E}_h, [\mathbb{E}_h', \theta]]) \cdot \theta) - \mathrm{Tr}_s(g(\mathcal{R}_h) \cdot \theta^2) \tag{3.36}$$

Lemma 3.16 together with equation (3.36) shows that

$$d^\mathcal{M}\mathrm{Tr}_s(g(\mathcal{R}_h)\theta) + \mathrm{Tr}_s(g(\mathcal{R}_h) \cdot \theta^2) \tag{3.37}$$
$$= d^X \mathrm{Tr}_s(g(\mathcal{R}_h; [\mathbb{E}_h', \theta]) \cdot \theta) - \mathrm{Tr}_s(g(\mathcal{R}_h; [\mathbb{E}_h', \theta])[\mathbb{E}_h, \theta])$$

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In equation (3.37), the left hand side is a 2-form on $\mathcal{M}$ with value in $\mathcal{G}^0$. For the right hand side of the equation, keeping in mind that $\mathbb{E}'_h$ increases exotic degree by -1 while $\mathbb{E}''$ increases it by 1, we can decompose $\mathrm{Tr}_s(g(\mathcal{R}_h; [\mathbb{E}'_h, \theta])[\mathbb{E}_h, \theta])$ as a sum of its $\mathcal{G}^{-2}$ and $\mathcal{G}^0$ components respectively as:

$$\mathrm{Tr}_s(g(\mathcal{R}_h; [\mathbb{E}'_h, \theta])[\mathbb{E}_h, \theta]) = \mathrm{Tr}_s(g(\mathcal{R}_h; [\mathbb{E}'_h, \theta])[\mathbb{E}'_h, \theta]) \tag{3.38}$$
$$+ \mathrm{Tr}_s(g(\mathcal{R}_h; [\mathbb{E}'_h, \theta])[\mathbb{E}'', \theta]) \tag{3.39}$$

Comparing the $\mathcal{G}^{-2}$ components in equation (3.37), we get equation (3.33). Taking its adjoint, we get (3.34). Finally if we compare the $\mathcal{G}^0$ components in equation (3.37), we get

$$d^{\mathcal{M}}\mathrm{Tr}_s(g(\mathcal{R}_h) \cdot \theta) + \mathrm{Tr}_s(g(\mathcal{R}_h) \cdot \theta^2) = \bar{\partial}^X \mathrm{Tr}_s(g(\mathcal{R}_h; [\mathbb{E}'_h, \theta]) \cdot \theta)$$
$$- \mathrm{Tr}_s(g(\mathcal{R}_h; [\mathbb{E}'_h, \theta])[\mathbb{E}'', \theta])$$

Adding the zero term $\mathrm{Tr}_s(g(\mathcal{R}_h; [\mathbb{E}'', \theta]) \cdot [\mathbb{E}'', \theta])$ to it, we get equation (3.32). $\qquad\square$

Lemma 3.18. *The last term* $\mathrm{Tr}_s(g(\mathcal{R}_h; [\mathbb{E}_h, \theta]) \cdot [\mathbb{E}'', \theta])$ *in equation (3.32) is given by:*

$$\mathrm{Tr}_s(g(\mathcal{R}_h; [\mathbb{E}_h, \theta]) \cdot [\mathbb{E}'', \theta]) = \partial^X \mathrm{Tr}_s(g(\mathcal{R}_h; [\mathbb{E}'', \theta]) \cdot \theta) \tag{3.40}$$
$$+ \mathrm{Tr}_s(g(\mathcal{R}_h; \theta) \cdot [\mathbb{E}'_h, [\mathbb{E}'', \theta]])$$

And the term $\mathrm{Tr}_s(g(\mathcal{R}_h; [\mathbb{E}'', \theta]) \cdot \theta)$ *satisfies:*

$$\bar{\partial}^X \mathrm{Tr}_s(g(\mathcal{R}_h; [\mathbb{E}'', \theta]) \cdot \theta) + \mathrm{Tr}_s(g(\mathcal{R}_h; \theta) \cdot [\mathbb{E}'', [\mathbb{E}'', \theta]]) = 0 \tag{3.41}$$

Proof. We consider the following differential and compute it as in lemma 3.16. Again we assume without loss of generality that $g(T) = T^n$.

$$d^X \sum_{i=1}^{n} \mathrm{Tr}_s(\mathcal{R}_h^{i-1}\theta\mathcal{R}_h^{n-i}[\mathbb{E}'', \theta]) \tag{3.42}$$

$$= \sum_{i+j+k=n-2} \mathrm{Tr}_s(\mathcal{R}_h^i[\mathbb{E}_h, \mathcal{R}_h]\mathcal{R}_h^j\theta\mathcal{R}_h^k[\mathbb{E}'', \theta]) + \sum_{i=1}^{n} \mathrm{Tr}_s(\mathcal{R}_h^{i-1}[\mathbb{E}_h, \theta]\mathcal{R}_h^{n-i}[\mathbb{E}'', \theta])$$

$$- \sum_{i+j+k=n-2} \mathrm{Tr}_s(\mathcal{R}_h^i\theta\mathcal{R}_h^j[\mathbb{E}_h, \mathcal{R}_h]\mathcal{R}_h^k[\mathbb{E}'', \theta]) - \sum_{i=1}^{n} \mathrm{Tr}_s(\mathcal{R}_h^{i-1}\theta\mathcal{R}_h^{n-i}[\mathbb{E}_h, [\mathbb{E}'', \theta]])$$

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Again the terms in the first and third summations are zero by Bianchi identity. So we get

$$\mathrm{Tr}_s(g(\mathcal{R}_h; [\mathbb{E}_h, \theta]) \cdot [\mathbb{E}'', \theta]) = d^X \mathrm{Tr}_s(g(\mathcal{R}_h; \theta) \cdot [\mathbb{E}'', \theta]) \tag{3.43}$$
$$+ \mathrm{Tr}_s(g(\mathcal{R}_h; \theta) \cdot [\mathbb{E}_h, [\mathbb{E}'', \theta]])$$

Substitute (3.43) into (3.32) in lemma 3.17, we have

$$d^{\mathcal{M}} \mathrm{Tr}_s(g(\mathcal{R}_h) \cdot \theta) + \mathrm{Tr}_s(g(\mathcal{R}_h) \cdot \theta^2) \tag{3.44}$$
$$= \bar{\partial}^X \mathrm{Tr}_s(g(\mathcal{R}_h; [\mathbb{E}'_h, \theta]) \cdot \theta) - d^X \mathrm{Tr}_s(g(\mathcal{R}_h; [\mathbb{E}'', \theta]) \cdot \theta)$$
$$- \mathrm{Tr}_s(g(\mathcal{R}_h; \theta) \cdot [\mathbb{E}_h, [\mathbb{E}'', \theta]])$$

Note again the left hand side is a 2-form on $\mathcal{M}$ with values in $\mathcal{G}^0$, we get the first equality by comparing $\mathcal{G}^0$ components and the second equality in by comparing $\mathcal{G}^2$ components. $\qquad\square$

Combining the formulas we proved so far, we are ready to derive the following main theorem.

Theorem 3.19. *Let $g(T)$ be a convergent power series in T, then the $\mathcal{M}$-directional derivative of $\mathrm{Tr}_s(g(\mathcal{R}_h) \cdot \theta)$ is given by the following formula:*

$$d^{\mathcal{M}} \mathrm{Tr}_s(g(\mathcal{R}_h) \cdot \theta) = \frac{1}{2} \bar{\partial}^X \mathrm{Tr}_s(g(\mathcal{R}_h; [\mathbb{E}'_h, \theta]) \cdot \theta) - \frac{1}{2} \partial^X \mathrm{Tr}_s(g(\mathcal{R}_h; [\mathbb{E}'', \theta]) \cdot \theta) \tag{3.45}$$

Proof. We write $\mathrm{Tr}_s(g(\mathcal{R}_h; \theta)[\mathbb{E}_h, [\mathbb{E}'', \theta]])$ as the sum of its $\mathcal{G}^0$ and $\mathcal{G}^2$ components

$$\mathrm{Tr}_s(g(\mathcal{R}_h; \theta)[\mathbb{E}'_h, [\mathbb{E}'', \theta]]) + \mathrm{Tr}_s(g(\mathcal{R}_h; \theta)[\mathbb{E}'', [\mathbb{E}'', \theta]]) \tag{3.46}$$

then by Jacobi identity and flatness of $\mathbb{E}''$, we have

$$[\mathbb{E}'', [\mathbb{E}'', \theta]] = \frac{1}{2}[[\mathbb{E}'', \mathbb{E}''], \theta] = 0 \tag{3.47}$$

Similarly, we compute

$$[\mathbb{E}'_h, [\mathbb{E}'', \theta]] = [[\mathbb{E}'_h, \mathbb{E}''], \theta] - [\mathbb{E}'', [\mathbb{E}'_h, \theta]] \tag{3.48}$$

By equation (3.33), we can add $0 = [\mathbb{E}'_h, [\mathbb{E}'_h, \theta]]$ to the above equation and note that $\mathcal{R}_h = [\mathbb{E}'_h, \mathbb{E}'']$, we have

$$[\mathbb{E}'_h, [\mathbb{E}'', \theta]] = [\mathcal{R}_h, \theta] - [\mathbb{E}_h, [\mathbb{E}'_h, \theta]] \tag{3.49}$$

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By lemma 3.15, we have $d^{\mathcal{M}}(\mathcal{R}_h) = [\mathbb{E}_h, [\mathbb{E}'_h, \theta]]$ and therefore we can rewrite the above formula as

$$[\mathbb{E}_h, [\mathbb{E}'', \theta]] = [\mathbb{E}'_h, [\mathbb{E}'', \theta]] = [\mathcal{R}_h, \theta] - d^{\mathcal{M}}\mathcal{R}_h \qquad (3.50)$$

so we have

$$\mathrm{Tr}_s(g(\mathcal{R}_h; \theta)[\mathbb{E}_h, [\mathbb{E}'', \theta]]) = \mathrm{Tr}_s(g(\mathcal{R}_h; \theta)[\mathcal{R}_h, \theta]) - \mathrm{Tr}_s(g(\mathcal{R}_h; \theta)d^{\mathcal{M}}\mathcal{R}_h) \qquad (3.51)$$

By the property of Tr_s and $\mathcal{R}_h$ has even total degree, we have

$$\mathrm{Tr}_s(g(\mathcal{R}_h; \theta)d^{\mathcal{M}}\mathcal{R}_h) = \sum_{i=1}^n \mathrm{Tr}_s(\mathcal{R}_h^{i-1}\theta\mathcal{R}_h^{n-i}d^{\mathcal{M}}\mathcal{R}_h) \qquad (3.52)$$

$$= \sum_{i=1}^n \mathrm{Tr}_s(\theta\mathcal{R}_h^{n-i}d^{\mathcal{M}}\mathcal{R}_h\mathcal{R}_h^{i-1})$$

$$= \mathrm{Tr}_s(\theta g(\mathcal{R}_h; d^{\mathcal{M}}\mathcal{R}_h))$$

Using this equation, we can rewrite the last term in equation (3.40) in lemma 3.18 as:

$$\mathrm{Tr}_s(g(\mathcal{R}_h; \theta)[\mathbb{E}'_h, [\mathbb{E}'', \theta]]) = \mathrm{Tr}_s(g(\mathcal{R}_h; \theta)[\mathcal{R}_h, \theta]]) - \mathrm{Tr}_s(g(\mathcal{R}_h; \theta)d^{\mathcal{M}}\mathcal{R}_h)$$

$$= \mathrm{Tr}_s(g(\mathcal{R}_h; \theta)[\mathcal{R}_h, \theta]]) + \mathrm{Tr}_s(g(\mathcal{R}_h; d^{\mathcal{M}}\mathcal{R}_h)\theta)$$

$$= \mathrm{Tr}_s(g(\mathcal{R}_h; \theta)[\mathcal{R}_h, \theta]]) + d^{\mathcal{M}}\mathrm{Tr}_s(g(\mathcal{R}_h)\theta) \qquad (3.53)$$

$$+ \mathrm{Tr}_s(g(\mathcal{R}_h)\theta^2)$$

Finally we plug the equation (3.53) into equation (3.44), after collecting terms, we get

$$2d^{\mathcal{M}}\mathrm{Tr}_s(g(\mathcal{R}_h) \cdot \theta) + 2\mathrm{Tr}_s(g(\mathcal{R}_h) \cdot \theta^2) = \bar{\partial}^X\mathrm{Tr}_s(g(\mathcal{R}_h; [\mathbb{E}'_h, \theta]) \cdot \theta) \qquad (3.54)$$

$$- \partial^X\mathrm{Tr}_s(g(\mathcal{R}_h; [\mathbb{E}'', \theta]) \cdot \theta)$$

$$- \mathrm{Tr}_s(g(\mathcal{R}_h; \theta)[\mathcal{R}_h, \theta])$$

We expand the last term $\mathrm{Tr}_s(g(\mathcal{R}_h;\theta)[\mathcal{R}_h,\theta])$ explicitly to get

$$\mathrm{Tr}_s(g(\mathcal{R}_h;\theta)[\mathcal{R}_h,\theta]]) = \sum_{i=1}^{n}\mathrm{Tr}_s(\mathcal{R}_h^{i-1}\theta\mathcal{R}_h^{n-i}\mathcal{R}_h\theta) - \sum_{i=1}^{n}\mathrm{Tr}_s(\mathcal{R}_h^{i-1}\theta\mathcal{R}_h^{n-i}\theta\mathcal{R}_h)$$

$$= \sum_{i=1}^{n}\mathrm{Tr}_s(\mathcal{R}_h^{i-1}\theta\mathcal{R}_h^{n-i+1}\theta) - \sum_{i=1}^{n}\mathrm{Tr}_s(\mathcal{R}_h^{i}\theta\mathcal{R}_h^{n-i}\theta)$$

$$= \mathrm{Tr}_s(\theta\mathcal{R}_h^n\theta) - \mathrm{Tr}_s(\mathcal{R}_h^n\theta^2) = -2\mathrm{Tr}_s(\mathcal{R}_h^n\theta^2)$$

$$= -2\mathrm{Tr}_s(g(\mathcal{R}_h)\cdot\theta^2) \tag{3.55}$$

Plug equation (3.55) into (3.54), we have

$$2d^{\mathcal{M}}\mathrm{Tr}_s(g(\mathcal{R}_h)\cdot\theta) = \bar{\partial}^X\mathrm{Tr}_s(g(\mathcal{R}_h;[\mathbb{E}_h',\theta])\cdot\theta) - \partial^X\mathrm{Tr}_s(g(\mathcal{R}_h;[\mathbb{E}'',\theta])\cdot\theta) \tag{3.56}$$

Divide both sides by 2, we get the desired formula. $\qquad\square$

We are now ready to define secondary Bott-Chern classes for cohesive modules with Hermitian structures. Recall $\mathcal{G} = \mathcal{G}^0$ is the space of (p,p) forms, let $\mathcal{G}'$ be the subspace of $\mathcal{G}$ defined by $\mathrm{Im}\partial^X + \mathrm{Im}\bar{\partial}^X$. We will define the secondary classes as elements in $\mathcal{G}^0/\mathcal{G}'\cap\mathcal{G}^0$.

Definition 3.20. Assume k_1, k_2 be two Hermitian metrics on a cohesive module E. For a convergent power series $f(T)$, we define the secondary Bott-Chern form $\tilde{f}(k_1, k_2)$ associated to k_1, k_2 as an element in $\mathcal{G}^0/\mathcal{G}'\cap\mathcal{G}^0$ with a representatives $\tilde{f}(k_1, k_2;\gamma)$ in $\mathcal{G}^0$ given by:

$$\tilde{f}(k_1, k_2;\gamma) = \int_{\gamma}\mathrm{Tr}_s(f'(\mathcal{R}_h)\cdot\theta)d\gamma \tag{3.57}$$

where $\gamma(t)$ is a curve on $\mathcal{M}$ that connects k_1 to k_2. The following proposition shows that this is well-defined.

Proposition 3.21. *The equivalence class of $\tilde{f}(k_1, k_2;\gamma)$ in $\mathcal{G}^0/\mathcal{G}'\cap\mathcal{G}^0$ is independent of γ.*

Proof. Let τ be a another path connecting k_1 to k_2, then by convexity of $\mathcal{M}$, the loop $\eta = \gamma - \tau$ is the boundary of a smooth 2-simplex σ in $\mathcal{M}$. By Stokes formula, we have

$$\tilde{f}(k_1, k_2;\gamma) - \tilde{f}(k_1, k_2;\tau) = \int_{\partial\sigma}\mathrm{Tr}_s(f'(\mathcal{R}_h)\cdot\theta)d\eta = \int_{\sigma}d^{\mathcal{M}}\mathrm{Tr}_s(f'(\mathcal{R}_h)\cdot\theta)d\sigma \tag{3.58}$$

By theorem 3.19, the integrand is an element in $\mathcal{G}'$, the result follows. $\qquad\square$

Corollary 3.22. *The primary characteristic form is related to the secondary form via the following equation:*

$$f(E, h_2) - f(E, h_1) = -\bar{\partial}^X \partial^X \tilde{f}(h_1, h_2; \gamma) \tag{3.59}$$

for any path γ that connects h_1 to h_2.

Proof. By the Bott-Chern formula (3.25), we have

$$f(E, h_2) - f(E, h_1) = \int_\gamma d^{\mathcal{M}} \mathrm{Tr}_s(f(\mathcal{R}_h)) d\gamma \tag{3.60}$$

$$= -\mathrm{Tr}_s\{\int_\gamma \bar{\partial}^X \partial^X f'(\mathcal{R}_h) \cdot \theta d\gamma\} \tag{3.61}$$

the result follows from the definition of $\tilde{f}(h_1, h_2; \gamma)$. $\qquad\square$

4 Invariance of Bott-Chern classes under quasi-isomorphism

4.1 Invariance under gauge transformation

In this section, we study the Bott-Chern forms under deformation of the cohesive structures. Unlike the situation of Hermitian metrics, the characteristic classes will depend on the cohesive structures. In fact, this is why we want to refine the characteristic classes to take value in Bott-Chern cohomology since their image in deRham cohomology only depend on the underlying topological complex vector bundle structure and therefore are independent of the cohesive structure.

Definition 4.1. For a $\mathbb{Z}$-graded Hermitian vector bundle $(E^\bullet, h)$, define $\mathcal{E}$ to be the space of $\bar{\partial}^X$-superconnections of total degree 1 and define $\mathcal{E}''$ to be subspace of $\mathcal{E}$ whose elements satisfy the flatness condition.

Definition 4.2. Recall the subspaces $\mathcal{G}^\bullet$ defined by exotic degrees in section 2. For each $k \in \mathbb{Z}$, we define $\mathcal{G}''^{,k}$ to be the subspace of $\mathcal{G}^k$ whose elements are forms of type $(0, \bullet)$. Similarly we define $\mathcal{G}'^{,k}$ to be the subspace of $\mathcal{G}^k$ whose elements are forms of type $(\bullet, 0)$.

Example 4.3. If $E^\bullet$ is concentrated in degree 0 and we set $\mathcal{A}$ to be the space of $\bar{\partial}^X$-connections and $\mathcal{A}''$ to be the subspace of $\bar{\partial}^X$-flat connections, then by Koszul-Malgrange theorem, $\mathcal{A}''$ is the space of holomorphic structures on E. If the underlying complex manifold X is a Riemann surface, then every $\bar{\partial}^X$-connection is automatically flat and $\mathcal{A}'' = \mathcal{A}$ is an affine space modeled on $\mathcal{A}^{0,1}(X, \mathrm{End}E)$.

We assume $\mathcal{E}''$ is non-empty and it has a marked point $\mathbb{E}''$ such that $E = (E^\bullet, \mathbb{E}'', h)$ is a Hermitian cohesive module. The following lemma is a straightforward consequence of the explicit construction of Chern superconnections in Proposition 2.17.

Lemma 4.4. *$\mathcal{E}$ is an affine space modeled on $\mathcal{G}''^{,1}$. $\mathcal{E}''$ is the subspace whose elements $\alpha'' \in \mathcal{G}''^{,1}$ satisfy the Maurer-Cartan equation:*

$$\mathbb{E}''(\alpha'') + \frac{1}{2}[\alpha'', \alpha''] = 0 \tag{4.1}$$

The $*$ operation interchanges $\mathcal{G}''^{,\bullet}$ and $\mathcal{G}'^{,-\bullet}$. If $\mathbb{E}$ is the Chern superconnection of the Hermitian cohesive module E, then the space of Chern superconnection is in one-to-one correspondence with $\mathcal{E}''$. The correspondence is given by:

$$\mathbb{F} \xrightarrow{\bar{\partial}^X-\text{component}} \mathbb{F}'' \tag{4.2}$$

whose inverse is given by:

$$\mathbb{E} + \alpha' + \alpha'' \xleftarrow{\alpha'=(\alpha'')^*} \mathbb{E}'' + \alpha'' \tag{4.3}$$

Definition 4.5. By the above lemma, the tangent space $T_{\mathbb{E}''}\mathcal{E}''$ of $\mathcal{E}''$ at a point $\mathbb{E}''$ can be identified with the space of solutions in $\mathcal{G}''^{,1}$ to the equation:

$$\mathbb{E}''(\alpha'') = 0 \tag{4.4}$$

In particular, the tangent space is a subspace in $\mathcal{G}''^{,1}$. We define a one form δ'' on $\mathcal{E}''$ with values in $\mathcal{G}''^{,1}$ by the formula:

$$\delta''(\alpha'') = \alpha'', \ \forall \alpha'' \in T_{\mathbb{E}''}\mathcal{E}'' \tag{4.5}$$

By duality, we also define the one form γ' on $\mathcal{E}''$ with values in $\mathcal{G}'^{,-1}$ by the formula:

$$\delta'(\alpha'') = (\alpha'')^* = \alpha' \tag{4.6}$$

Finally, we set $\delta = \delta' + \delta''$ to be an one form with values in $\mathcal{G}''^{,1} \oplus \mathcal{G}'^{,-1}$.

Lemma 4.6. *If $\mathbb{E}_t$ is a family of Chern superconnections such that $\mathbb{E}''|_{t=0} = \mathbb{E}''$ and $\mathbb{E}''_t = \mathbb{E}'' + A''_t$, then the deformation of the curvatures $\mathcal{R}_t$ is given by:*

$$\frac{d}{dt}\mathcal{R}_t = [\mathbb{E}_t, \frac{d}{dt}\mathbb{E}_t] \tag{4.7}$$

where $\mathbb{E}_t t$ is the Chern superconnection of $(E^\bullet, \mathbb{E}''_t, h)$. Using the one form δ defined above, we can rewrite this formulas as

$$d^{\mathcal{E}}\mathcal{R} = -[\mathbb{E}, \delta] \tag{4.8}$$

Proof. We write $\alpha_t = \alpha'_t + \alpha''_t$ for the tangent vectors $\frac{d}{dt}\mathbb{E}_t$ for short. For each t, α''_t satisfies:

$$[\mathbb{E}''_t, \alpha''_t] = 0 \tag{4.9}$$

and by duality, α'_t satisfies

$$[\mathbb{E}'_t, \alpha'_t] = 0 \tag{4.10}$$

By definition, $\mathcal{R}_t = [\mathbb{E}''_t, \mathbb{E}'_t]$, so if we take its derivative, we have

$$\frac{d}{dt}\mathcal{R}_t = [\alpha''_t, \mathbb{E}''_t] + [\alpha'_t, \mathbb{E}'_t] \tag{4.11}$$

Adding up the above equations, we get the desired equality. $\qquad\square$

Proposition 4.7. *Let $f(T)$ be a convergent power series in T, the $\mathcal{E}$-directional differential of the Bott-Chern characteristic form $f(E, \mathbb{E}'', h)$ at $\mathbb{E}'' \in \mathcal{E}''$ is given by:*

$$d^{\mathcal{E}}f(\mathcal{R}) = -\mathrm{Tr}_s f'(\mathcal{R} \cdot [\mathbb{E}, \delta]) = -d^X \mathrm{Tr}_s f'(\mathcal{R} \cdot \delta) \tag{4.12}$$

Proof. The first equality is a direct consequence of previous lemma. The second equality follows from Bianchi identity $[\mathbb{E}, \mathcal{R}] = 0$ as before. $\qquad\square$

Corollary 4.8. *We compare the $\mathcal{G}^\bullet$ components of both sides of the equality (4.12). We get the following identities:*

$$d^{\mathcal{E}}\mathrm{Tr}_s f(\mathcal{R}) = -\partial^X \mathrm{Tr}_s(f'(\mathcal{R}) \cdot \delta'') - \bar{\partial}^X \mathrm{Tr}_s(f'(\mathcal{R}) \cdot \delta') \tag{4.13}$$

$$\partial^X \mathrm{Tr}_s(f'(\mathcal{R}) \cdot \delta') = 0 \tag{4.14}$$

$$\bar{\partial}^X \mathrm{Tr}_s(f'(\mathcal{R}) \cdot \delta'') = 0 \tag{4.15}$$

Equation (4.13) is the first transgression formula over $\mathcal{E}''$. Motivated by the results we established in previous section, equation (4.14) and (4.15) are expected to admit a further transgression. However this is not the case in general for otherwise we would have proved $\mathrm{Tr}_s f(\mathcal{R})$ is even independent of the cohesive structure. We will show the terms $\mathrm{Tr}_s(f'(\mathcal{R}) \cdot \delta')$ and $\mathrm{Tr}_s(f'(\mathcal{R}) \cdot \delta'')$ admit a further transgression formula when restricted to certain subspaces in $\mathcal{E}''$.

Definition 4.9. We define the generalized Dolbeault complex associated to a cohesive module E by $(\mathcal{G}''^{,\bullet}, \mathbb{E}'')$. By the flatness condition on $\mathbb{E}''$, it defines a differential. The cohomology of the complex is defined to be the Dolbeault cohomology of the cohesive module E.

Remark 4.10. The Euler characteristic $\chi(E)$ of the Dolbeault cohomology group of a cohesive module E is studied in a different paper [Qia16]. By Hodge theorey, $\chi(E)$ is given by the index of a Dirac-type operator D^E defined by $\sqrt{2}(\mathbb{E}'' + (\mathbb{E}'')^*)$ where $(\mathbb{E}'')^*$ is the formal adjoint of $\mathbb{E}''$ with respect to a Hermitian metric h_E on E and a Hermitian metric h_X on X. It is proved that the index is computed by the classical Atiyah-Singer index formula:

$$\mathrm{Ind}(D^E) = (2\pi i)^{-\dim X} \int_X \mathrm{Todd}(X) \cdot \mathrm{ch}(E) \tag{4.16}$$

where $\mathrm{Todd}(X)$ is the Todd genus and $\mathrm{ch}(E)$ is the Chern character form associated to the power series $\exp(-T)$.

Definition 4.11. Let $\mathbb{E}''_t$ be a one parameter family of cohesive structures. The tangent vectors $\alpha''_t = \dot{\mathbb{E}}''_t$ satisfies $\mathbb{E}''_t(\alpha''_t) = 0$ so they are pointwise closed with respect to the generalized Dolbeault operator $\mathbb{E}''_t$. We say the family is exact if there exist a smooth section γ''_t with values in $\mathcal{G}''^{,0}$ such that

$$[\mathbb{E}''_t, \gamma''_t] = \delta''_t, \forall t \tag{4.17}$$

That is to say, the tangent vectors α''_t defines zero cohomology class in the first Dolbeault cohomology groups and in addition, we can find a smooth lift of them.

Example 4.12. Consider the group $\mathrm{GL}(E)$ whose elements are of the form $f = \sum_{k=0}^{\dim X} f_k$ where $f_k \in \mathcal{A}^{0,k}(X, \mathrm{End}^{-k} E)$ and f_0 is invertible. Since $\mathcal{A}^{\bullet>0}$

is nilpotent and we required f_0 to be invertible, $\mathrm{GL}(E)$ forms a group. $\mathrm{GL}(E)$ acts on $\mathcal{E}$ and preserves the subspace $\mathcal{E}''$ via gauge transformation:

$$(\mathbb{E}'')^f = f^{-1} \circ \mathbb{E}'' \circ f = f^{-1} \circ [\mathbb{E}'', f] + \mathbb{E}'' \tag{4.18}$$

If f_t is a one parameter family of gauge group elements in $\mathrm{GL}(E)$ such that $f_0 = \mathrm{Id}_E$, then for any choice of $\mathbb{E}'' \in \mathcal{E}''$, we claim the one parameter family of cohesive structures $\mathbb{E}''_t = (\mathbb{E}'')^{f_t}$ is exact.

To see this, we simply take $\gamma''_t = f_t^{-1}\frac{d}{dt}f_t$ with values in $\mathcal{G}''^{,1}$ and we can compute

$$\frac{d}{dt}\mathbb{E}''_t = \frac{d}{dt}(f_t^{-1} \circ [\mathbb{E}'', f_t] + \mathbb{E}'') \tag{4.19}$$

$$= -\gamma''_t \circ f_t^{-1} \circ [\mathbb{E}'', f_t] + f_t^{-1} \circ [\mathbb{E}'', f_t \circ \gamma''_t] \tag{4.20}$$

$$= -\gamma''_t \circ \mathbb{E}''_t + \gamma''_t \circ \mathbb{E}'' + \mathbb{E}''_t \circ \gamma''_t - \gamma''_t \circ \mathbb{E}'' \tag{4.21}$$

$$= [\mathbb{E}''_t, \gamma''_t] \tag{4.22}$$

This shows that the family $\mathbb{E}''_t$ obtained by applying a family of gauge transformations is exact.

Remark 4.13. If E is again just a holomorphic vector bundle concentrated in degree 0, then $\mathrm{GL}(E)$ is just the group of invertible linear automorphisms. Two holomorphic vector bundle structure on E are equivalent if and only if they differ by a gauge transformation.

Definition 4.14. If $\mathcal{S}$ is a submanifold of $\mathcal{E}''$ for which the restriction δ'' is exact and admits a smooth lift $\gamma'' \in \mathcal{G}''^{,0}$ such that $[\mathbb{E}'', \gamma''] = \delta''$, we say $\mathcal{S}$ is exact.

Proposition 4.15. *If $\mathcal{S}$ is exact with a lift γ'' of δ'' and we set $\gamma' = (\gamma'')^*$, then we have the following identities:*

$$\bar{\partial}^X \mathrm{Tr}_s(f'(\mathcal{R} \cdot \gamma'')) = \mathrm{Tr}_s(f'(\mathcal{R}) \cdot \delta'') \tag{4.23}$$

$$\partial^X \mathrm{Tr}_s(f'(\mathcal{R} \cdot \gamma')) = -\mathrm{Tr}_s(f'(\mathcal{R}) \cdot \delta') \tag{4.24}$$

Proof. By Bianchi identity, we have

$$d^X \mathrm{Tr}_s(f'(\mathcal{R} \cdot \gamma'')) = \mathrm{Tr}_s(f'(\mathcal{R}) \cdot [\mathbb{E}, \gamma'']) \tag{4.25}$$

We compare the $\mathcal{G}^2$ component of the equation and we get the first equation (4.23). If we replace γ'' by $\gamma' = (\gamma'')^*$, it satisfies

$$[\mathbb{E}', \gamma'] = -\delta' \tag{4.26}$$

by taking adjoints of $[\mathbb{E}'', \gamma''] = \delta''$. The same argument as above shows the second equation (4.24). □

Combining Proposition 4.7 and Proposition 4.15, we have the following double transgression formula for Bott-Chern forms over an exact submanifold.

Proposition 4.16. *With the same assumption as above, if we set $\gamma = \gamma' + \gamma'' \in \mathcal{G}^0$, the $\mathcal{S}$-directional differential of Bott-Chern forms is given by:*

$$d^{\mathcal{S}}\mathrm{Tr}_s f(\mathcal{R}) = -\partial^X \bar{\partial}^X \mathrm{Tr}_s(f'(\mathcal{R}) \cdot \gamma) \tag{4.27}$$

As a corollary, the Bott-Chern characteristic classes remains invariant over S. We shall now apply this proposition to prove the invariance of Bott-Chern cohomology classes under homotopy equivalences.

For two cohesive modules $(E^\bullet, \mathbb{E}'')$ and $(F^\bullet, \mathbb{F}'')$, we have the following criteria for homotopy equivalence. For its proof, we refer to [Blo10].

Proposition 4.17. $(E^\bullet, \mathbb{E}'')$ *and* $(F^\bullet, \mathbb{F}'')$ *are equivalent in the homotopy category* $\mathrm{Ho}(\mathcal{P}_\mathcal{A})$ *if and only if there is a degree zero closed morphism* $\phi \in \mathcal{P}_\mathcal{A}^0$ *which is a quasi-isomorphism between the complexes* $(E^\bullet, \mathbb{E}_0'')$ *and* $(F^\bullet, \mathbb{F}_0'')$.

Using this criteria, the invariance of Bott-Chern classes will be proved in two steps. First we show that if the underlying complex $(E^\bullet, \mathbb{E}_0'')$ is acyclic, the characteristic classes are zero. Next we show that the Bott-Chern characteristic classes are additive with respect to short exact sequences of mapping cones and reduce to the acyclic case.

4.2 Transgression formula for acyclic complex

We start by define a rescaling operation on superconnections. If $t \in \mathbb{R}^+$ is a positive real constant and $\mathbb{E}$ is a superconnection, define

$$\mathbb{E}_t = \sum_k t^{1-k}\mathbb{E}_k \tag{4.28}$$

It's easy to verify that $\mathbb{E}_t'' = (\mathbb{E}'')_t$ and $\mathbb{E}_t''$ are flat. So $\mathbb{E}_t''$ forms a smooth family of cohesive structures.

Definition 4.18. We define the grading operator N_E for the cohesive module E as an element in $\mathcal{A}^0(X, \mathrm{End}^0(E^\bullet))$ such that it acts by:

$$N_E(A \otimes \omega) = |A| \cdot A \otimes \omega \tag{4.29}$$

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Lemma 4.19. *With N_E so defined, we choose $\gamma_t'' = \frac{1}{t} N_E \in \mathcal{G}^0$ and we have*

$$[\mathbb{E}_t'', \gamma_t''] = \frac{1}{t} \sum_k (1-k)(\mathbb{E}_{k,t}'') = \frac{d}{dt} \mathbb{E}_t'' \tag{4.30}$$

Proof.

$$\frac{d}{dt} \mathbb{E}_t'' = \sum_k (1-k) t^{-k} \mathbb{E}_k'' = \frac{1}{t} \sum_k (1-k)(\mathbb{E}_k'')_t \tag{4.31}$$

On the other hand, we may take $s \in E^d$ so we have

$$\mathbb{E}_t'' \circ N_E(s) = d \sum_k t^{1-k} \mathbb{E}_k''(s) \tag{4.32}$$

and note that $\mathbb{E}_k''(s) \in \mathcal{A}^{0,k}(X, E^{d+1-k})$, so we have

$$N_E \circ \mathbb{E}_{k,t}''(s) = N_E \sum_k t^{1-k} \mathbb{E}_k''(s) = \sum_k t^{1-k} (d+1-k) \mathbb{E}_k''(s) \tag{4.33}$$

Taking the difference of the above two equalities and divide both sides by t, we get (4.30). $\qquad\square$

Since N_E is clearly self adjoint, we have $\gamma' = \gamma'' = \frac{1}{t} N_E$ and we have the following corollary.

Corollary 4.20. *For the family of rescaled cohesive structures, we have*

$$\frac{d}{dt} \mathrm{Tr}_s(e^{-\mathcal{R}_t}) = -\frac{2}{t} \partial^X \bar{\partial}^X \mathrm{Tr}_s(e^{-\mathcal{R}_t} \cdot N_E) \tag{4.34}$$

Finally, by the same arguments in [BGS88a], if $(E^\bullet, \mathbb{E}_0'')$ is acyclic, then the degree zero component Δ_E of its curvature

$$\mathcal{R}_t^{0,0} = t^2 (\mathbb{E}_0'' + \mathbb{E}_0')^2 = t^2 \Delta_E$$

is a strictly positive element and the characteristic form of the Chern character

$$\mathrm{Tr}_s \exp(-R_t) = \mathrm{Tr}_s \exp(-t^2 \Delta_E + \mathcal{O}(t))$$

decays exponentially fast uniformly on X when t approaches ∞. Applying this result and let $t \to \infty$, we transgressed the characteristic forms to zero as desired.

Theorem 4.21. *If $(E^\bullet, \mathbb{E}'')$ is a cohesive module such that $(E^\bullet, \mathbb{E}''_0)$ is an acyclic complex, then the integral*

$$\mathcal{I}_E = \int_1^\infty \mathrm{Tr}_s(\exp(-\mathcal{R}_t) \cdot N_E) \frac{dt}{t} \tag{4.35}$$

is finite and

$$\mathrm{Tr}_s \exp(-\mathcal{R}) = \partial^X \bar{\partial}^X \mathcal{I}_E \tag{4.36}$$

4.3   Additive properties of Bott-Chern classes

We begin by defining the mapping cone of a closed morphism. Let $\phi \in \mathcal{P}^0_{\mathcal{A}}(E, F)$ be a closed degree 0 morphism between two cohesive modules $(E^\bullet, \mathbb{E}'')$ and $(F^\bullet, \mathbb{F}'')$.

Definition 4.22. The mapping cone $\mathrm{Cone}(\phi)$ is the cohesive module $(\mathrm{Cone}(\phi), \mathbb{C}''_\phi)$ whose underlying complex vector bundle is defined by:

$$\mathrm{Cone}(\phi)^\bullet = F^\bullet \oplus E^{\bullet+1}$$

and with respect to this decomposition, the cohesive structure $\mathbb{C}''_\phi$ is given by:

$$\mathbb{C}''_\phi = \begin{pmatrix} \mathbb{F}'' & \phi \\ 0 & -\mathbb{E}'' \end{pmatrix}$$

Consider the one parameter family of morphisms $\phi_t = t\phi$ with $t \in [0, 1]$ such that $\phi_0 = 0$ and $\phi_1 = \phi$. The corresponding mapping cones have the same underlying bundle $\mathrm{Cone}(\phi)^\bullet$ and cohesive structures $\mathbb{C}''_t$. It's a simple calculation that

$$\frac{d}{dt} \mathbb{C}''_{\phi_t} = \begin{pmatrix} 0 & \phi \\ 0 & 0 \end{pmatrix} = \alpha'' \tag{4.37}$$

is constant. If we set for $t > 0$, $\gamma''_t \in \mathcal{G}''^{,0}$ by the formula:

$$\gamma''_t = \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{t} \cdot \mathrm{Id}_F \end{pmatrix} = \frac{1}{t}\gamma''$$

then it satisfies that

$$[\mathbb{C}''_t, \gamma''_t] = \frac{d}{dt} \mathbb{C}''_t \tag{4.38}$$

So we can apply the previous Proposition 4.16 for all nonzero value of t. Note again in this case $\gamma' = \gamma''$, we have the following equality of characteristic forms.

$$f(\text{Cone}(\phi), \mathcal{C}_1'') - f(\text{Cone}(\phi), \mathcal{C}_s'') = -\partial^X \bar{\partial}^X [\int_s^1 2\text{Tr}_s f'(\mathcal{R}_t) \cdot \gamma'' \frac{dt}{t}] \quad (4.39)$$

It's clear that we can't let $t \to 0$ in the above formula due to the singularity. Instead, motivated by the construction in [BC65], we will modify the integrand to remove the singularity.

To do this, we calculate $\mathcal{R}_t$ explicitly as

$$\mathcal{R}_t = \begin{pmatrix} \mathcal{R}_F + t^2 \phi\phi^* & t(\mathbb{F}'\phi - \phi\mathbb{E}') \\ t(\phi^*\mathbb{F}'' - \mathbb{E}''\phi^*) & \mathcal{R}_E + t^2 \phi^*\phi \end{pmatrix} = \mathcal{R}_0 + tA_t \quad (4.40)$$

where $\mathcal{R}_0$ is the curvature computed by $\mathbb{C}_0''$ computed using the direct sum Hermitian form

$$\mathcal{R}_0 = \begin{pmatrix} \mathcal{R}_F & 0 \\ 0 & \mathcal{R}_E \end{pmatrix} \quad (4.41)$$

and A_t is the reminder term

$$A_t = \begin{pmatrix} t\phi\phi^* & \mathbb{F}'\phi - \phi\mathbb{E}' \\ \phi^*\mathbb{F}'' - \mathbb{E}''\phi^* & t\phi^*\phi \end{pmatrix} \quad (4.42)$$

If we evaluate a convergent power series $g(T)$ on $\mathcal{R}_t$, we have

$$g(\mathcal{R}_t) = g(\mathcal{R}_0 + tA_t) = g(\mathcal{R}_0) + tR_g(t, \mathcal{R}_0, A_t) \quad (4.43)$$

for some reminder term R_g that is a power series in t with coefficients that are polynomials in $\mathcal{R}_0$ and A_t.

We have already shown that $\text{Tr}_s f(\mathcal{R}_0)$ is d^X-closed by Proposition 2.24, therefore it is also $\partial^X \bar{\partial}^X$-closed. So we can subtract it from the above equation and derive the following equation:

$$f(\text{Cone}(\phi), \mathcal{C}_1'') - f(\text{Cone}(\phi), \mathcal{C}_s'') = -2\partial^X \bar{\partial}^X \int_s^1 (f'(\mathcal{R}_t)\gamma'' - f'(\mathcal{R}_0)\gamma'') \frac{dt}{t}$$
$$(4.44)$$

Now we substitute equation (4.43) with $g = f'$ into the above equality, the singular term $\frac{1}{t}$ cancels out and we get

$$f(\text{Cone}(\phi), \mathcal{C}_1'') - f(\text{Cone}(\phi), \mathcal{C}_s'') = -2\partial^X \bar{\partial}^X \int_s^1 R_{f'}(t, A_t, \mathcal{R}_0) dt \quad (4.45)$$

Now the we can let $s \to 0$ since the integrand is bounded and we established the following proposition.

Proposition 4.23. *The characteristic form $f(\mathrm{Cone}(\phi), \mathcal{C}_1'')$ coincide with $f(\mathrm{Cone}(\phi), \mathcal{C}_0'')$ in Bott Chern cohomology.*

Under the direct sum Hermitian metric $h_E \oplus h_F$ on the mapping cone $\mathrm{Cone}^\bullet$, we have the following simple equality of differential forms:

$$f(\mathrm{Cone}(\phi)^\bullet, \mathbb{C}_0'', h_\phi) = f(F^\bullet, \mathbb{F}'', h_F) - f(E^\bullet, \mathbb{E}'', h_E) \qquad (4.46)$$

where the minus sign comes from the shift in degree of $E^\bullet$ in the mapping cone. Together with the previous corollary, we established the following proposition.

Proposition 4.24. *The Bott Chern cohomology classes of an exact triangle*

$$0 \to (E^\bullet, \mathbb{E}'') \xrightarrow{\phi} (F^\bullet, \mathbb{F}'') \xrightarrow{i_F} (\mathrm{Cone}^\bullet(\phi), \mathbb{C}_\phi)$$

is additive in the sense

$$f(E^\bullet, \mathbb{E}'') - f(F^\bullet, \mathbb{F}'') + f(\mathrm{Cone}^\bullet(\phi), \mathbb{C}_\phi) = 0 \qquad (4.47)$$

in Bott-Chern cohomology

If ϕ is a homotopy equivalence, it was shown in [Blo10] that this is equivalent to $\phi_0 : (E^\bullet, \mathbb{E}_0'') \to (F^\bullet, \mathbb{F}_0'')$ being a quasi-isomorphism. Therefore the mapping cone $\mathrm{Cone}(\phi)$ is acyclic so proposition 4.21 shows that the left side of equation (4.46) is zero in Bott-Chern cohomology. Combining these, we established our main result below.

Theorem 4.25. *If two cohesive modules $(E^\bullet, \mathbb{E}'')$ and $(F^\bullet, \mathbb{F}'')$ are homotopy equivalent, then they have the same Bott-Chern cohomology classes.*

As a corollary, we can define characteristic classes for an object $\mathscr{S}$ in $D_{\mathrm{Coh}}^b(X)$ as follows. We choose a cohesive module representative $(E^\bullet, \mathbb{E}'')$ and equip it with some Hermitian structure h_E. Then for any convergent power series $f(T)$, we define the associated class $f(\mathscr{S})$ as the class of $f(E, h_E)$. In particular, this extends Bott-Chern cohomology classes to coherent sheaves.

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